- Operator splitting
- Stability of explicit schemes
- Explicit operator splitting for Navier-Stokes equations

Operator splitting:

Lets say that we the following

$$u_t = Au + Bu$$

For example

$$A = u_x$$

$$B = \nu u_{xx}$$

Can we then split the scheme for the composed operator $A+B$ such that we solve

first

$$u_t = Au$$

then

$$u_t = Bu$$

Yes! For each time-step we split.

Hence

$$\frac{u^{n+\frac{1}{2}} - u^n}{2\Delta t} = Au^n$$

$$\frac{u^{n+1} - u^{n+\frac{1}{2}}}{2\Delta t} = Bu^{n+1/2}$$

Stability of explicit schemes

Consider

1) $u_t = C u_x$

2) $u_t = D u_{xx}$

Explicit schemes are

1) $$\frac{u_i^{n+1} - u_i^n}{\Delta t} = C \, \frac{u_i^n - u_{i-1}^n}{\Delta t}$$

2) $$\frac{u_i^{n+1} - u_i^n}{\Delta t} = D \, \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

Why do we call it an explicitt scheme?

$\Longrightarrow$ We have an explicitt formula, no need to solve a linear system

Stability requirement :

1) $\Delta t \leq C h$

2) $\Delta t \leq D h^2$

[These are rough guidelines valid for finite difference in 1D, some adjustemnt ~~are~~ are needed for FEM in 3D]

What about Navier-Stokes?

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right) = -\nabla p + \mu \nabla^2 \vec{v} + \vec{f}$$

$$\nabla\cdot\vec{v}$$

How about an explicit scheme?

Let us plug in:

1)

$$\rho\left(\frac{\vec{v}^{\,n+1} - \vec{v}^{\,n}}{\Delta t} + (\vec{v}^{\,n}\cdot\nabla)v^n\right) = -\nabla p^n + \mu \nabla^2 \vec{v}^{\,n} + \vec{f}^{\,n}$$

What is the trouble here?

1. No update for the pressure because there is no time-derivate on p

4. No idea whether $\nabla\cdot V^{n+1} = 0$

Let us modify such that

1.) and 2.) are addressed

$\Rightarrow$

2)
$$\rho\left(\frac{v^{n+1}-v^{n}}{\Delta t}+(\vec{v}^{n}\cdot\nabla)v^{n}\right)=-\nabla p^{n+1}+\nu\nabla^{2}\vec{v}^{n}+\vec{f}^{n}$$

$$\nabla\cdot\vec{v}^{n+1}=0$$

All hell breaks loose !!

The scheme is no longer explicit and we have to deal with the constraint $\nabla\cdot v^{n+1}=0$ and an additional variable.

Let us call

$\vec{V}^{n+1}$ in 1) V^*

and subtract 1) ~~as~~ from 2)

$\Longrightarrow$ $$p^{n+1} - p^{n} = \phi$$

$$\rho \frac{\left(\vec{V}^{n+1} - \vec{V}^{*}\right)}{\Delta t} = \overbrace{-\nabla p^{n+1} + \nabla p^{n}}$$

$$\nabla \cdot V^{n+1} = 0$$

$\Longrightarrow$ $$\rho \left(\frac{\vec{V}^{n+1} - \vec{V}^{*}}{\Delta t} \right) = -\nabla \phi \qquad (*)$$

$$\nabla \cdot \vec{V}^{n+1} = 0$$

Taking $\nabla \cdot$ of $(*)$

Gives $$-\nabla^2 \phi = -\frac{\nabla \cdot \vec{V}^{*}}{\Delta t}$$

We arrived at a "operator splitting", often called a projection scheme.

1) Compute $\vec{v}^*$ according to

$$\rho\left(\frac{\vec{v}^* - \vec{v}^n}{\Delta t} + (\vec{v}^n \cdot \nabla)\vec{v}^n\right) = -\nabla p^n + \mu \nabla^2 \vec{v}^n + f^n$$

2) Compute ϕ as

$$-\nabla^2 \phi = -\frac{\nabla \cdot \vec{v}^*}{\Delta t}$$

[We notice that this step is not explicit]

3) Compute $p^{n+1} = p^n + \phi$

4) Compute $\vec{v}^{n+1} = \vec{v}^* - \frac{\Delta t}{\rho} \nabla \phi$