Lecture 5

~~The~~ Lax - Milgram theorem

Remember the strong form:

$$-\mu \Delta u + w \cdot \nabla u = f \quad \text{in } \Omega$$

$$u = g \quad \text{on } \partial\Omega$$

And the weak form:

Find $u \in H^1_g(\Omega)$ such that

$$a(u,v) = L(v) \quad \forall H^1_0(\Omega) \qquad 1)$$

$$a(u,v) = \int_\Omega \mu \nabla u \cdot \nabla v + w \cdot \nabla u \, v \, dx$$

$$L(v) = \int_\Omega f v \, dx$$

Lax-Milgram's theorem

Let V be a Hilbert-space, $a(\cdot,\cdot)$ a bilinear form and $b(\cdot)$ a linear form, both on V. Then

The problem: Find $u \in V$ such that

$$a(u,v) = b(v) \quad \forall\, v \in V$$

is <u>well-posed</u> if

3 conditions are met

1) $a(u,u) \geq \alpha \|u\|_V \quad \forall\, u \in V$

2) $a(u,v) \leq C \|u\|_V \|v\| \quad \forall\, u,v \in V$

3) $b(v) \leq D \|v\|_V \quad \forall\, v \in V.$

What does it mean

in a linear algebra setting ?

$Ax = b$, $A \in \mathbb{R}^{n,n}$, $b \in \mathbb{R}^n$

1) $x^T A x > 0 \quad \forall \; x \in \mathbb{R}^n$

2) $x^T A y < \infty \quad \forall \; x, y \in \mathbb{R}^n$

3) $b^T y < \infty \quad \forall \; y \in \mathbb{R}^n$

Here 2) and 3) are easy

to verify for A, b —

if neither A nor b has an entry

that is ∞. Then we are ok.

1) is more tricky and less intuitive

and as we will see the challenging one.

Let us try to verify the conditions

3)

$$L(v) \leq D \|v\|_V$$

For us $V = H_0^1(\Omega)$

Hence

$$L(v) = \int_\Omega f v \, dx$$

$$\leq \left(\int_\Omega f^2 dx\right)^{1/2} \left(\int v^2 dx\right)^{1/2}$$

Cauchy-Schwitz

$$= \|f\|_{L^2} \|v\|_{L^2}$$

$$\leq \|f\|_{L^2} \|v\|_{H_0^1}$$

$$D =$$

$\boxed{\text{Perform Lifting}}$

~~Condition~~

Then 2)

pull out

$$a(u,v) = \int_\Omega \nu \nabla u \cdot \nabla v \, dx + \int_\Omega (w \cdot \nabla u)\, v \, dx$$

$$\leq \nu \, \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} + \|v\|_{L^\infty} \|\nabla u\| \|v\|$$

$$\leq \nu \, \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} + \|v\|_{L^\infty} \|\nabla u\| \underbrace{C \|\nabla v\|}_{\text{Poincare}}$$

$$= (\nu + C \|w\|_{L^\infty} C) \, |\nabla u|_{L^2} |\nabla v|_{L^2}$$

We know that on H_0^1

$|\nabla u|_{L^2}$ is a norm

Finally, the <u>hard part</u>

$$a(u,u) \geq \underbrace{\alpha \|u\|_V}_{\alpha \|\nabla u\|_{L^2}}$$

First notice that we can write

$$a(u,v) = b(u,v) + c_w(u,v)$$

with $$b(u,v) = \int_\Omega \mu \nabla u \cdot \nabla v \, dx$$

$$c_w(u,v) = \int_\Omega (w \cdot \nabla u) \, v \, dx$$

Clearly

$$b(u,u) = \int \mu (\nabla u)^2 dx$$

$$= \mu \|\nabla u\|_{L^2}^2$$

Hence, for $b(u,u)$ we have positivity / coersivity with $\alpha = \mu$.

What about $c_w(u,v)$?

A main trick here is to show that $c_w(\cdot,\cdot)$ is skew-symmetric if $\nabla\cdot\vec{w} = 0$

<u>Integration by parts</u>

$$c_w(u,v) = \int_\Omega w\cdot\nabla u \, v \, dx$$

$$= -\int_\Omega w\cdot\nabla v \, u \, dx - \underbrace{\int \nabla\cdot w \, u \, v \, dx}_{\text{incompressible}}$$

$$+ \int_\Gamma u v w\cdot n \, ds = 0$$

Dirichlet conditions and lifting

Hence

$$C_w(u,v) = \int_\Omega w \cdot \nabla u \, v \, dx$$

$$= -\int w \nabla v \, u \, dx = C_w(v,u)$$

$\Rightarrow$ skew-symetry

Furthermore, letting $v = u$

$\Rightarrow$

$$C_w(u,u) = -C_w(u,u)$$

Hence $C_w(u,u) = 0$ $\square$

0

This means that

$$\alpha = \nu$$

$$C = \nu + C_\Omega |w|_\infty$$

Lax-Milgram gives us stability in the sense that

$$\|u\|_{H^1_0} \le \frac{C}{\alpha} \underbrace{\|f\|_{H^{-1}}}_{\le \|f\|_{L^2}}$$

Hence, if ν is small and $|w|_\infty$ large very little stability is gained.