Lecture 4 : chap 4

An a priori error estimate

Given a finite element method with polynomials of order m

Then ~~the~~ we have an error estimate

$$\left(\int_\Omega (\nabla(u - u_h))^2 \, dx \right)^{1/2} \leq \frac{k_1}{k_0} C h^{m-1} |u|_m$$

We remember Bramble - Hilbert lemma

$$|u - P_m|_{k,p} \leq C h^{m-k} |u|_{m,p} \qquad 1)$$

Here $p = 2$. and

$$|u|_{m,2} = |u|_m$$

and in the Exercises tomorrow we will show equivalences of various norms and inner products:

E.g

2) $$k_0 \int_\Omega (\nabla u)^2 dx \leq \int_\Omega (k \nabla u) \nabla u \, dx \leq k_1 \int_\Omega (\nabla u)^2 dx$$

$$k_0 = \inf_{x \in \Omega} k(x) \; , \; k_1 = \sup_{x \in \Omega} k(x).$$

The proof is simple,

but we need to use

<u>Galerkin orthogonality</u>

$$k_0 \int_\Omega \left(\nabla(u-u_n)\right)^2 dx \overset{2)}{\leq} \int_\Omega \left(k\nabla(u-u_n)\right)\cdot\left(\nabla(u-u_n)\right) dx$$

$$\leq \int_\Omega \left(k\nabla(u-u_n)\right)\cdot\left(\nabla(u-P_m u + P_m u - u_n\right) dx$$

notice now that

$(P_m u - u_n) \in V_n$

and $k(\nabla(u-u_n))$ is

orthogonal

$$\leq \int_\Omega \left(k\nabla(u-u_n)\right)\cdot\left(\nabla(u-P_m u)\right) dx$$

Take out k

$$\le k_1 \int_\Omega \nabla(u-u_h)\cdot\nabla(u-P_m u)\,dx$$

Hence

$$k_0 \int(\nabla(u-u_h))^2\,dx \le k_1 \int_\Omega \nabla(u-u_h)\,\nabla(u-P_m u)\,dx$$

$$\underset{\text{Cauchy-Schwarz}}{\le} k_1\left(\int(\nabla(u-u_h))^2\,dx\right)^{1/2} \underbrace{\left(\int(\nabla(u-P_m u))^2\,dx\right)^{1/2}}_{\le\, C h^{m-1}|u|_m}$$

~~Bramble-Hilbert~~

In other words

$$\left(\int (\nabla(u-u_h))^2\,dx\right)^{1/2} \le \frac{k_1}{k_0}\cancel{\left(\int (\nabla(u-u_h))\,dx\right)^{1/2}}\, C h^{m-1}|u|_m$$

[End chapter with and example that shows this in practice.] !

Chapter 4

Exactly the same analysis as in the previous chapter ~~[illegible]~~ reveals shortcomings in ~~[illegible]~~ numerical ~~[illegible]~~ methods of today.

Consider

$$-\nu \Delta u + \vec{v} \cdot \nabla u = f \quad \text{in } \Omega$$

$$u = g \quad \text{on } \partial\Omega.$$

For $|\vec{v}| \gg \mu$ the problem is a singular perturbation problem giving rise to exponential behaviour near ~~the~~ (part of the) boundaries and oscillations in many numerical schemes. A main topic of this chapter is to explain why!

Consider the simplified 1D

problem

$$- u_x - \mu u_{xx} = 0$$

$$u(0) = 0$$

$$u(1) = 1$$

The analytical solution is

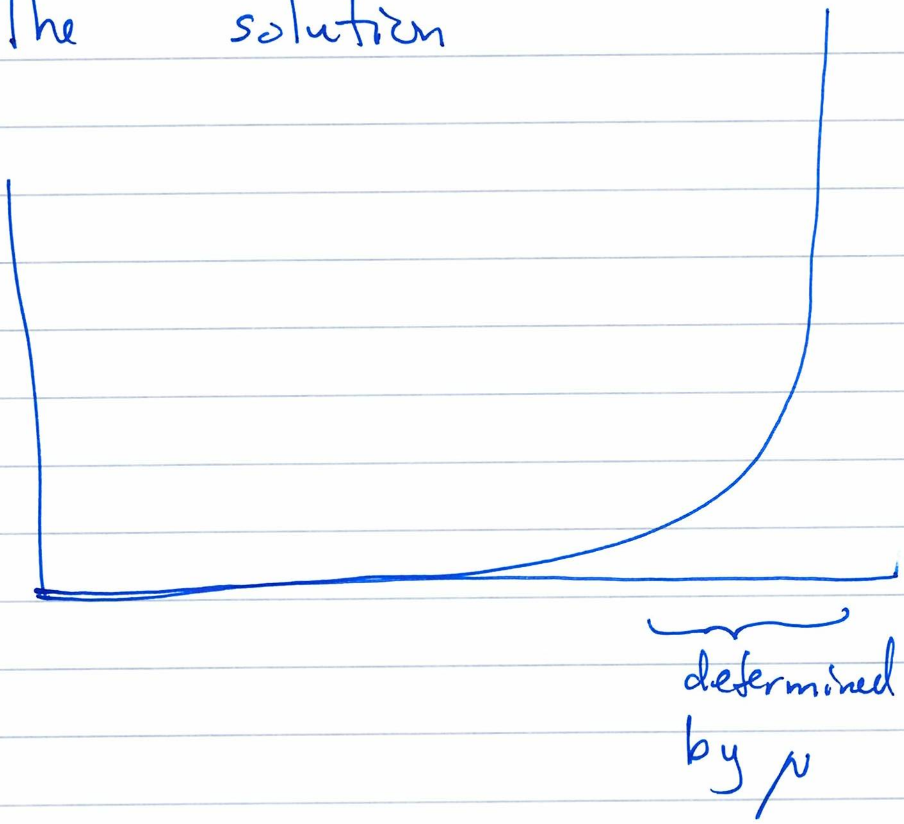
$$u(x) = \frac{e^{-x/\mu} - 1}{e^{-1/\mu} - 1}$$

We can easily check that

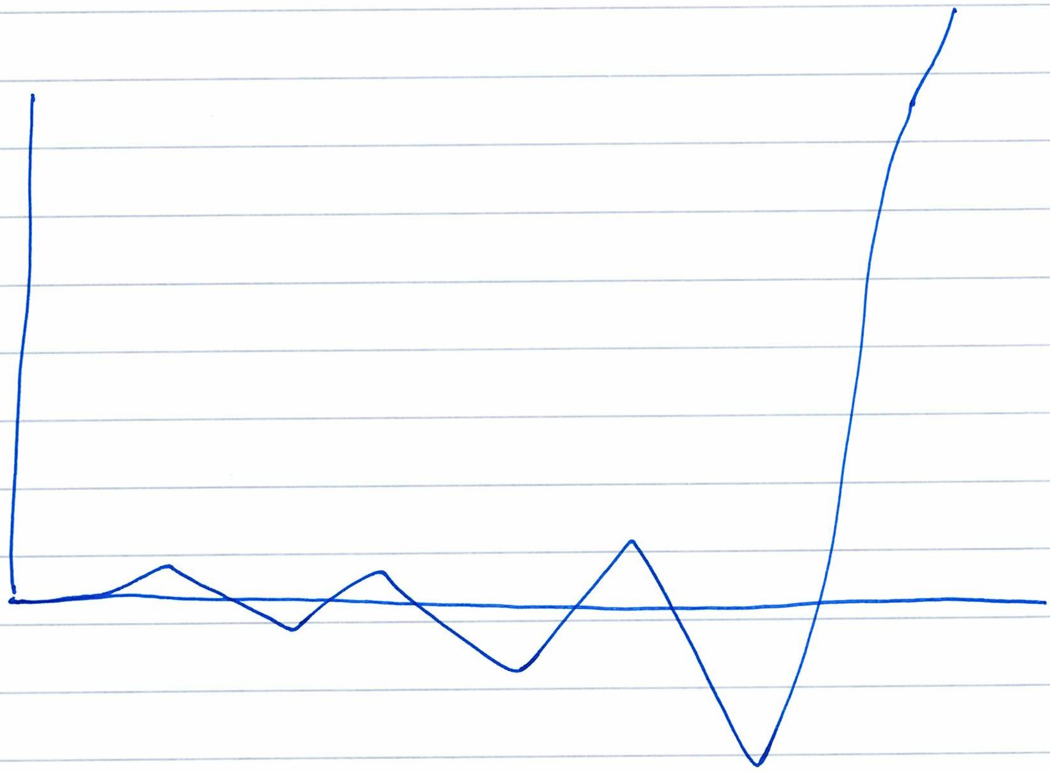
both the PDE and BC

are satisfied.

The solution



The numerical solution



Let us then consider

a finite difference scheme

$$-\frac{\nu}{h^2}\left[u_{i+1} - 2u_i + u_{i-1}\right] - \frac{v}{2h}\left[u_{i+1} - u_{i-1}\right] = 0$$

$$u_0 = 0 \quad , \quad u_N = 1$$

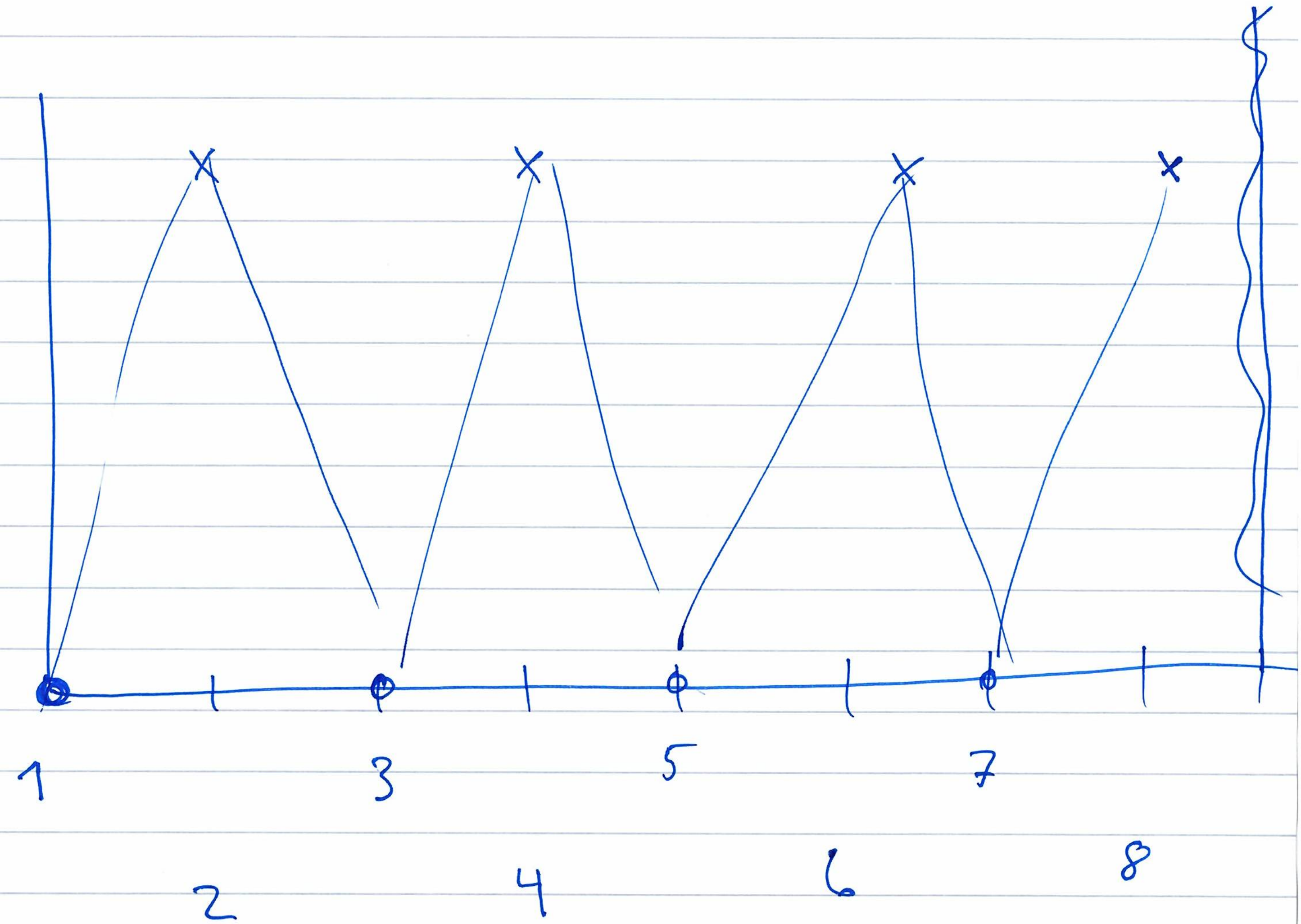
we use central differences

because they give 2 - order

convergence

Notice here that for the case $\nu = 0$ we have

$$-\frac{V}{2h}\left[u_{i+1} - u_{i-1}\right] = 0$$



Hence, with 9 nodes, we get wild oscillations. We would like to connect neighbours

<u>Upwinding</u> connects neighbours but is only first order.

$$v < 0 \quad : \quad \frac{du}{dx}(x_i) = \frac{1}{h}\left(u_{i+1} - u_i\right)$$

$$v > 0 \quad : \quad \frac{du}{dx}(x_i) = \frac{1}{h}\left(u_i - u_{i-1}\right)$$

<u>Upwinding removes oscillations</u>!

Upwinding can be seen as central differences with artificial diffusion ~~of 2 order~~ scaled by $\frac{h}{2}$

(Hence a first order perturbation)

The calculation is simple:

central scheme $\dfrac{u_{i+1} - u_{i-1}}{2h}$

+ artificial diffusion $\dfrac{h}{2}\left(\dfrac{-u_{i+1} + 2u_i - u_{i-1}}{h^2}\right)$

= upwind $\dfrac{u_i - u_{i-1}}{h}$

Streamline-diffusion Petrov-Galerkin methods

Let us again start with a ~~continuous~~ weak formulation of the ~~problem~~ continuous problem.

We 1. multiply by a test function v and integrate

2. Use Gauss-Green's lemma

3. Use boundary conditions.

We arrive at

Find $u \in H^1_g(\Omega)$ such that

$$a(u, w) = b(w) \quad \forall\ w \in H^1_0(\Omega)$$

where $a(u,w) = \int_\Omega \mu \nabla u \cdot \nabla w \, dx + \int_\Omega v \cdot \nabla u \, w \, dx$

$$b(w) = \int_\Omega f w \, dx$$

A standard Galerkin method is hence:

Find $u_h \in V_{h,g}$ such that

$$a(u_h, w_h) = b(w_h) \quad \forall\, w_h \in V_{h,0}$$

We may add artificial diffusion to the problem by eg solving

$$a(u_h, w_h) + \frac{h}{2}(\nabla u_h, \nabla w_h) = (f, w_h)$$

$$\forall\, w_h \in V_{h,0}$$

The scheme with artificial diffusion will be consistent, i.e. the truncation error tends to zero, but the

Petrov-Galerkin is strongly consistent (truncation error is zero for every discretization)

The Petrov-Galerkin method modifies the test functions.

If N_j is the test function of Galerkin

then $L_j = N_j + \beta h (w \cdot \nabla N_j)$

is the test function of P-G.

We notice that we have the same number of test functions so we will have an equal number of unknowns and equations!

The matrix of Galerkin

$$A_{ij} = a(N_i, N_j) = \int_\Omega \nu \nabla N_i \cdot \nabla N_j$$

$$+ \int (w \cdot \nabla N_i)\, N_j$$

P-G

$$A_{ij} = a(N_i, L_j) = \int_\Omega \nu \nabla N_i \cdot \nabla N_j$$

$$+ \int_\Omega (w \cdot \nabla N_i)\, N_j$$

what we want (consistently added numerical diffusion in streamline direction)

$$+ \cancel{\beta h \int_\Omega (w \cdot \nabla N_i)} \;\; \beta h \int (w \cdot \nabla N_i)\, w \cdot \nabla N_j)$$

bad term, third order

$$+ \beta h \int \nu \nabla N_i \cdot \nabla (w \cdot \nabla N_j))$$