Chap 3, Lecture 3

The finite element method for elliptic equations – precise formulation

Strong formulation:

Find $u \in C^2(\Omega)$ such that

$$-\nabla \cdot (k \nabla u) = f \quad \text{in } \Omega$$

$$u = g$$

$$k \frac{\partial u}{\partial n} = h$$

We arrive at the weak form

by 1) multiplying with a test function and integrating over the domain

2) Use Gauss-Green's lemma

3) Use the boundary condition

[Did it in detail on the first lecture]

We arrive at the weak formulation

Given $f \in H_D^{-1}(\Omega)$, $h \in H^{-1/2}(\partial\Omega_N)$

Find $u \in H^1_{g,D}(\Omega)$ such that

$$\int_\Omega (k \nabla u) \cdot \nabla v = \int_\Omega f v \, dx + \int_{\partial\Omega_N} h v \, ds$$

$$\forall \quad v \in H^1_{0,D}(\Omega)$$

Here

$$H^1_{0,D}(\Omega) = \{ u \in H^1(\Omega) \mid Tu = 0 \text{ on } \partial\Omega_D \}$$

$$H^1_{g,D}(\Omega) = \{ u \in H^1(\Omega) \mid Tu = g \text{ on } \partial\Omega_D \}$$

In order to define a finite element method, let Ω_h be a mesh consisting of a set E_h of cells. We assume that they are simplices (triangles in 2D, tetrahedrons in 3D)

Simplices fit well with standard polynomials.

~~Wat~~ What do I mean by that:

consider P_k the space of polynomials

5)

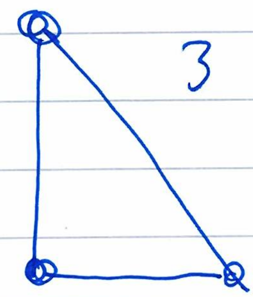
$P_1 = \{1, x, y\}$ number of basis : 3

$P_2 = \{1, x, y, xy, x^2, y^2\}$: 6

$P_3 = \{1, x, y, xy, x^2, y^2, x^3, x^2y, y^2x, y^3\}$: 10

$P_4 = \{\ldots x^4, x^3y, x^2y^2, x^1y^3, y^4\}$: 15

Points :



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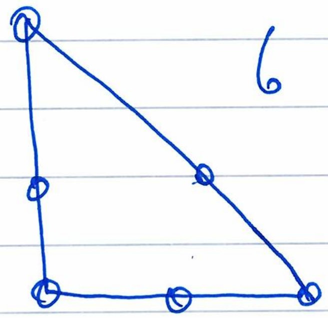
P_1



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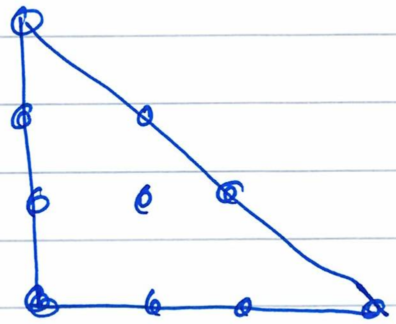
P_2



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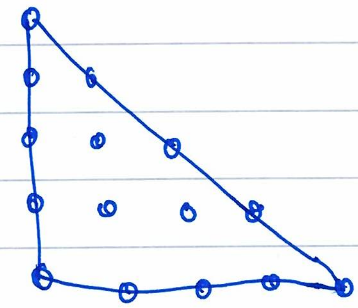
P_3



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There are many ways to represent the polynomial space P_k

- monomial / power basis
- Berstein polynomials
- Jacobi polynomials
- Lagrange polynomials.

Lagrange is very useful. the basis is set up such that the basis is 1 in one of the points shown and zero elsewhere.

Hence, the basis functions are set up such that

$$N_i(x_j) = \delta_{ij}$$

3.2)

What is an elliptic partial differential equation?

The equation is strickly positive with respect to some inner product.

We need several concepts to show positivity.

1) Lifting
2) Poincare
3) strickt positivity of k.

Poincaré's inequality

for every $u \in H^1_{0,D}(\Omega)$ we have

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}$$

the inequality can be motivated by a Taylor series

$$|f(x+h)| = |f(x)| + h \cancel{f'(x)} \max_{x \leq x^* \leq x+h} |f'(x^*)|$$

if $f(x)$ is zero, which we know it is at certain parts of the boundary ($f \in H^1_{0,D}(\Omega)$)

$$\Rightarrow |f(x+h)| \leq h \max_{x^*} |f'(x^*)|$$

In $H^1_{g,D}(\Omega)$ we do not know that u is 0 at the boundary as g can be anything.

Hence, we need to transform our problem to a problem with 0 at the boundary !!

This is called <u>lifting</u>

The trick is easy, used in analysis but not typically in implementation.

Let u solve ~~the stro~~

(I write it on strong form for simplicity)

$$-\nabla\cdot(k\nabla u) = f \quad \text{in } \Omega$$

$$u = g \quad \text{on } \partial\Omega_D$$

$$k\frac{\partial u}{\partial n} = h \quad \text{on } \partial\Omega_N$$

Let then $u = G + u_0$ where $G|_{\partial\Omega_D} = g$ but can be anything elsewhere. Then u_0 solves the problem

$$-\nabla\cdot(k\nabla u_0) = f + \nabla\cdot(k\nabla G)$$

$$u_0 = 0$$

$$k\frac{\partial u_0}{\partial n} = h - k\frac{\partial G}{\partial n}$$

Hence u_0 solves a problem with homogenous boundary conditions on part of the boundary.

The positivity of k.

We need that for every x in Ω

$$k_0 \xi^2 \leq \xi^T k(x) \xi \leq k_1 \xi^2 \qquad \forall \xi \in \mathbb{R}^n$$

3.3 An error estimate

Galerkin orthogonality

Consider the numerical error

$e_h = u - u_h$.

We have

$$\int_\Omega (k\nabla u)\cdot(\nabla v)\,dx = \int_\Omega f\,v\,dx + \int_{\partial\Omega_N} h\,v\,ds \qquad 1)$$

and

$$\int_\Omega (k\nabla u_h)\cdot\nabla v_h\,dx = \int_\Omega f\,v_h\,dx + \int_{\partial\Omega_N} h\,v_h\,ds \qquad 2)$$

since $V_{h,g} \subset H^1_{g,D}$

we may in 1) choose $v = v_h$

Then subtracting 2) from 1) we get

$$\int_\Omega (k\nabla(u-u_h)\cdot v_h\,dx = \int_\Omega (f-f)\,v_h\,dx$$

$$+ \int_{\partial\Omega_N} (h-h)\,v_h\,ds$$

$$= 0$$

Hence the error $e_h = u - u_h$

is orthogonal to all $v_h \in V_{h,g}$

with respect to the $\int k\nabla(\cdot)(\nabla\cdot)$ inner product

$$\boxed{\begin{array}{l}\text{Result from the exercise} \\ k_0 \int_\Omega (\nabla u)^2\, dx \le \int (k\nabla u)\nabla u \le k_1 \int_\Omega (\nabla u)^2\, dx\end{array}}$$

Cea's lemma:

$$k_0 \int (\nabla(u-u_h))^2 dx \le$$

$$\int (k\nabla(u-u_h))\nabla(u-u_h)\, dx \le$$

$$\int k\nabla(u-u_h)\cdot\nabla(u - P_m u \cancel{+ P_m u - u_h})\, dx$$

$P_m u$ and $u_h \in V_{g,h}$

$$= \int k\nabla(u-u_h)\cdot\nabla(u-P_m u)$$

$$\le k_1 \|\nabla(u-u_h)\| \|\nabla(u-P_m u)\|$$

we already had a bound on $\|\nabla(u - P_m u)\| \leq C h^{m-1} \|u\|_m$

$\Rightarrow$

$$\|u - P_m u\|_1 \leq C h^{m-1} \|u\|_m$$

$\Rightarrow$

$$\|\nabla(u - u_h)\| \leq \frac{k_1}{k_0} C h^{m-1} \|u\|_m$$

m is the polynomial order of the elent space.