Chapter 2, lecture 2

- Sobolev spaces – crash course

- Explain the motivation behind the course – something between ~~cors~~ courses in functional analysis and viscous flow

- Want to finalize a book: Feedback is welcome

- Current shape
Theory first then more about Navier - Stokes.

- What is a norm ?
- What is an inner product ?
- What extra does an inner product give ?

We will look more into the definitions of norms and inner products in the exercises.

<u>But</u> remark that inner products bring geometry. !!

A function may be <u>ortogonal</u> to another

L^p norms

$L^p(\Omega)$:

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \, dx \right)^{1/p}$$

$$L^p(\Omega) = \left\{ u \mid \|u\|_{L^p(\Omega)} < \infty \right\}$$

Let $\Omega = (0,1)$ then the Sobolev space

$W_k^p(\Omega)$ has norm

$$\|u\|_{p,k} = \int_0^1 \sum_{i \le k} \left(\left| \frac{\partial^i u}{\partial x^i} \right| \right)^p dx \Big)^{1/p}$$

$$W^{p,k} = \{ u \mid \|u\|_{p,k} < \infty \}$$

Hence the Sobolev space

$W^{p,k}(\Omega)$ consists of all functions in which the k'th derivative can be integrated to the power p. on Ω.

We considered a 1D geometry but the extension to any dimension is straightforward.

A semi-norm

(What is the difference between a semi-norm and a norm?)

$$|u|_{p,k} = \left(\int_\Omega \sum_{i \leq k} \left| \frac{\partial^i u}{\partial x^i} \right|^p dx \right)^{1/p}$$

Why the need for semi-norms?

It gives sharper estimates (potentially) and simpler analysis.

~~Consider the~~

Consider the two equations

1)
$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

2)
$$\begin{aligned} -\Delta u + \beta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Omega. \end{aligned}$$

with $\beta > 0$.

The first equation only have terms with two derivatives

The second equation have terms with <u>two</u> and zero derivatives.

Hence, the first equation fits with analysis based on the semi-norm.

The second fits with analysis based on norms.

Inner products $(p = 2)$

$$(u, v)_k = \sum_{i \le k} \int_\Omega \frac{\partial^i u}{\partial x^i} \frac{\partial^i v}{\partial x^i} \, dx$$

Examples

Norms of $\sin(k\pi x)$ on $\Omega = (0,1)$

$$\|u\|_{L^2(\Omega)} = \left(\int_0^1 \left(\sin(k\pi x)\right)^2 dx \right)^{1/2}$$

$$= \sqrt{\frac{1}{2}} \quad \text{for any } k.$$

$$\|u\|_{H^1(\Omega)} = \left((k\pi)^2 \int_0^1 \left(\cos(k\pi x)\right)^2 dx \right)^{1/2}$$

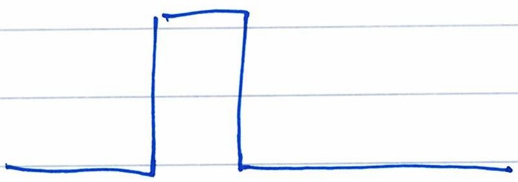
$$= \sqrt{\frac{1}{2}}\; k\pi$$

The norm increases linearly in the frequency of the oscillations $\square$

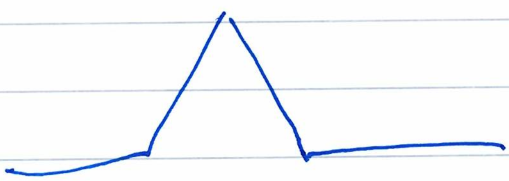
One can also compute L^7 or other norms: they behave similar to L^2.

Examples of functions

1.



in L^2 (or generally L^p)



in L^2 (L^p) and H^1.

Polynomial approximation in Sobolev spaces.

What about universial approximation theorem for neural networks ?

" There exists networks that will approximate a continous function well "

More spesific :

$f \in C(X, Y)$ then

for any ε there exists a

<u>shallow</u> network such that

$$\sup_{x \in X} \| f(x) - g(x) \| < \varepsilon$$

$$g(x) = C \sigma (Ax + b)$$

[σ should not be a polynomial]

What does Taylor ~~sy~~ say ?

$$\left| f(x+h) - P_{n,k} f(x) \right| \leq O\left(h^{k+1}\right)$$

$$P_{n,k} f(x) = f(x) + \sum_{n=1}^{k} \frac{f^{n}(x)}{n!} h^{n}$$

this is certainly more spesific, constructive and sharp than the universal approximation theorem.

That said, there are many bad things with Taylor series from a practical point of view

1. It can only be used in a small neigbourhood of x
2. It scales bad.

In the finite element method, we exploit small domains (the elements) $\boxed{\text{with } h < 1.}$

Here we have the corresponding estimate

$$\boxed{|u - P_m u|_{k,p} \leq C h^{m-k} |u|_{m,k}}$$

k and m are here the number of derivatives.

Sometimes people talk about first order or second order schemes.

What does it refer to

$\longrightarrow$ The polynomial approximation

$P_m \rightarrow m$

$\longrightarrow h^{m-k}$, i.e $m-k$. ?

I'd say $m-k$ but many will say m.

Standard question for instance at a MSc defence.

Eigenvalues and ~~eigenvectors~~ Eigenfunctions

Eigenvalues and eigenvectors are important :

1. Provide intuition
2. Extensively used in solid mechanics applications
3. Help us define negative and fractional norms

4. Are computed wrongly with the finite element method if you do not know what you are doing .

It is well known that for

$-\Delta$ on the unit interval $(0,1)$

with homogenous Dirichlet conditions

the eigenvalues and eigenfunctions

are

$$\lambda_k = (\pi k)^2$$
$$e_k = \sin(k\pi x)$$

However, if we compute

the eigenvalues of the stiffnessmatrix

of the finite element method,

i.e

$$A_{ij} = \int_\Omega \nabla N_i \cdot \nabla N_j \, dx$$

The eigenvalues look different.

The finite element method introduces a mesh dependent scaling because it is a <u>variational method</u>.

To explain it, note that a function f may be represented as two different vectors:

1. $f \approx \sum_j f_j N_i$ (nodal)

2. $b_i = \int_\Omega f N_i$ (dual)

The mass matrix transforms the ~~used~~ nodal representation to the dual

$$M f = b$$

$$M_{ij} = \int N_i N_j$$

$$b_i = \int f N_i$$

Hence, for the finite element method we need to solve the generalized eigenvalue problem

$$A x = \lambda M x$$

to get the real eigenvalues.

Negative and fractional norms

Given a matrix A that is SPD. Then we may define A^q in terms of ~~eigen~~ the spectral decomposition

$$A = Q \Lambda Q^{-1}$$

$$A^q = Q \Lambda^q Q^{-1}$$

For example, if we look
~~inte~~ into a PDE book
we will find that
for

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

We will have the a priori
estimate

$$\cancel{\|u\|_1} \quad |u|_1 = \|f\|_{-1}$$

what does it mean ?

Lets just see what happens if u where an eigenfunction.

$$u_k = \sin(k\pi x) \quad , \quad \Omega = (0,1)$$

$$|u_k|_1 = \frac{\pi k}{\sqrt{2}}$$

what if we would like to compute

$$\| u_k \|_{-1} \quad ?$$

$$|u_k|_0 = \frac{1}{\sqrt{2}}$$

$$|u_k|_{-1} = \frac{1}{\sqrt{2}} \frac{1}{\pi k}$$

$$|u_k|_2 = \frac{1}{\sqrt{2}} (\pi k)^2 \qquad |u_k|_s = \frac{1}{\sqrt{2}} \left((\pi k)^2 \right)^s$$

$\boxed{\text{Crazy}}$

Consider two solutions

(homogenous Dirichlet)

$$-\Delta u = f \quad \Big| \quad u = (-\Delta)^{-1} f$$

$$-\Delta v = g \quad \Big| \quad v = (-\Delta)^{-1} g$$

Then $(u, v)_1 = (f, g)_{-1}$

$$\begin{gathered} \| \\ (-\Delta u, v)_0 \\ \| \\ (f, v)_0 \\ \| \\ (f, (-\Delta)^{-1} g)_0 \end{gathered}$$