EXAM FOR MEK4250 2023

Introduction

Five problems/topics are given for this exam. For each problem, the candidate must prepare a 20 minutes oral presentation. Try to communicate a good overview and understanding of the topic, but compose the talk so that you can demonstrate knowledge about details too. The student is expected to be able to stick to one subject for the 30 minutes for top grades. There are no aids besides a whiteboard and this document with the exam problems (experience with this type of exam and various aids tells that learning the content by heart gives by far the best delivery that demonstrates solid understanding). We will throw a die and the number of eyes determines the topic to be presented. After your presentation, you will be given some questions, either about parts of your presentation or facts from the other topics. After each presentation, the next candidate can throw the die and thereby get about 10 minutes to collect the thoughts before presenting the assigned topic.

1. The finite element method

Explain Ciarlet´s definition of a finite element. Explain the concept of functionals and function spaces. How are degrees of freedom used to ensure that the finite element spaces are part of certain function spaces? Show that a finite element may conveniently be definied in terms of a reference element. List common elements in common spaces.

2. Weak formulation and finite element error estimation

Formulate a finite element method for the Poisson problem with a variable coefficient $\kappa : \Omega \to \mathbb{R}^{d\times d}$. Assume that κ is positive and symmetric. Show that Lax-Milgram's theorem is satisifed. Consider extensions to e.g. convection-diffusion equation and the elasticity equation. Derive *a priori* error estimates in terms of Cea's lemma for the finite element method in the energy norm. Describe how to perform a estimation of convergence rates.

3. Discretization of convection–diffusion

Derive a proper variational formulation of the convection–diffusion problem. Derive sufficient conditions that make the problem well-posed. Discuss why oscillations appear for standard Galerkin methods and show how SUPG methods resolve these problems. Discuss also approximation properties in light of Cea's lemma.

4. Discretization of Stokes

Derive a proper variational formulation of the Stokes problem. Discuss the four Brezzi conditions that are need for a well-posed continuous problem. Explain why oscillations might appear in the pressure for some discretization techniques. Present expected approximation properties for mixed elements that satisfy the inf-sup condition, and discuss a few examples like e.g. Taylor–Hood, Mini, and Crouzeix–Raviart. Discuss also how one might circumvent the inf-sup condition by stabilization.

5. Discretization of Navier–Stokes

Explain the difference between operator splitting and algebraic splitting in the context of the incompressible Navier–Stokes equations. We remark that algebraic splitting is a term usually used for discretizations where the PDEs are discretized in space prior to time. Show disadvantages for operator splitting schemes associated with boundary conditions. Explain the advantage with operator splitting schemes.

6. Other formulations.

Consider the various formulations of the Poisson problem. Show that minimization of energy $\int_\Omega 1/2(\nabla u)^2 - fudx$ corresponds to a weak formulation that through Green´s lemma gives a strong formulation. Do the same thing for the least square formulation and arrive at a bi-harmonic equation with some additional boundary conditions. Consider both the mixed formulation and least square formulation of the mixed formulation, and note that only the first requires the Brezzi conditions.