Lecture 9

Discretizing in space prior to time.

Our favorite equations:

$$\rho\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right) = -\nabla p + \mu \Delta u + f \quad 1)$$

$$\nabla\cdot u = 0 \quad 2)$$

+ BC and IC

2)

Discretizing in space

$\longrightarrow$

$$u_h = \sum_i u_i(t)\, N_i(x)$$

$$p_h = \sum_j p_j(t)\, L_j(x)$$

$\Rightarrow$ N_i, L_j are sets of concrete basis functions on our meshes!

$\Rightarrow$ $u_i(t)$ and $p_j(t)$ are continuous functions and have not been discretized yet.

Let us start with a weak formulation of the Navier-Stokes equations. Multiply 1) and 2) with test functions, integrate over the domain and do integration by parts for those terms that seem to need it.

4)

Start with 1) : multiply with test function

$$\rho\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right)\cdot v = (-\nabla p + \mu\Delta u + f)\cdot v$$

integrate over the domain Ω

$$\int_\Omega \rho\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right)\cdot v = \int_\Omega (-\nabla p + \mu\Delta u + f)\cdot v$$

Integration by parts

$$\int_\Omega (-\nabla p + \mu\Delta u + f)\cdot v = \int_\Omega p \operatorname{div} v \, dx$$

$$- \int_\Omega \mu \nabla u : \nabla v \, dx$$

$$= g \quad + \int_{\partial\Omega} \left(p - \frac{\partial u}{\partial n}\right) \cdot v \, ds$$

Comment:

- Integration by parts is sometimes done also for the $\int (u\cdot\nabla)u\cdot v$ term.

- The term $\int (u\cdot\nabla)u\, v$ is non-linear.

- The term $\int (u\cdot\nabla)u\, v$ messes up the Brezzi conditions. Showing them or similar conditions relates to the

$\boxed{\text{Millennium problem.}}$

Weak formulation

Find u, p such that

$$\int_\Omega \left(\rho \left(\frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) \cdot v + \mu \nabla u : \nabla v - p \cdot \nabla \cdot v \right) dx$$

$$= \int_\Omega f \cdot v + \int_{\partial \Omega} g v \, ds$$

$$\int_\Omega \nabla \cdot u \; q = 0$$

$\forall \; v, q$ and $\forall \; t \in [0, T]$

Alternatively, Find u, p

$$\int_0^T \int_\Omega \left(\rho \left(\frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) + \mu \nabla u : \nabla v - p \nabla \cdot v \right) dx$$

$$= \int_0^T \int_\Omega f v + \int_0^T \int_{\partial\Omega} g v$$

$$\int_0^T \int_\Omega \nabla \cdot u \; q = 0$$

$\forall \quad v, q$ ~~...~~

[not oxactly the same as on the previous page - in strickt math]

Using the formulation on page 6),

$$u_h = \sum_i u_i(t)\, N_i(x) \,, \quad p_h = \sum_j p_j(t)\, L_j(x)$$

$$v = N_k(x) \,, \quad q = L_\ell(x)$$

$$\Longrightarrow$$

Find u_h, p_h such that

$$\int_\Omega \Big(\rho \Big(\frac{\partial u_h}{\partial t} + (u_h \cdot \nabla) u_h \Big) N_k(x) + \mu \nabla u_h : \nabla N_k$$

$$- p_h \nabla \cdot N_k \Big) dx = \int_\Omega f \cdot N_k \, dx + \int_{\partial\Omega} g\, N_k \, ds$$

Let then

$$M_{ij} = \int_{\Omega} N_i N_j$$

$$K(u)_{ij} = \int_{\Omega} (u_h \cdot \nabla) N_j N_i$$

$$A_{ij} = \int_{\Omega} \nu \nabla N_i : \nabla N_j$$

$$Q_{ij} = \int \nabla \cdot N_i \cdot L_j$$

Then our system looks like

$$1)\quad M\dot{u}_n + K(u_n)\,u_n = -Qp_n + Au_n + b$$

$$Q^T u_n = 0$$

We have a non-linear system of ODEs (1) coupled to a set of algebraic equations (2)

$\Longrightarrow$ Differential-algebraic system of non-linear equations.

The system

10) - 1) is hard to

solve efficiently (!!!!)

It is - non-linear

- non-symmetric

- indefinite

$\Longrightarrow$ worst kind.

Let us consider the approach taken earlier :

1. Compute a tentative velocity
2. Compute a pressure related variable (ϕ)
3. Project the velocity into something that has ($\nabla \cdot u = 0$) by using the ϕ
4. Update pressure.

An explicit scheme

1.

$$Mu^* = Mu^l + \Delta t(-K(u^l)u^l - Qp^l + Au^l + f^l$$

We assume

$$u^{l+1} = u^* + u^c$$

What is u^c ?