Lecture 8

Three things to discuss today:

1. inf-sup

2. boundary conditions for splitting/projection schemes with Navier-Stokes

3. Going from non-homogeneous Dirichlet conditions to homogenous Dirichlet conditions
   $\longrightarrow$ lifting operations

Invertibility of $BA^{-1}B^T$ can be ensured if A^{-1} is invertible and B has full rank.

- B has full rank would imply that BB^T is invertible.

- Notice that B^TB will <u>not</u> ~~have full~~ be invertible.

3)

B is $m \times n$ $\quad n \gg m$

kernel of $B^T = 0$

(means that B^T has full rank.)

is equivalent to

1) $\max_v (v, B^T p) > 0 \quad \forall p$

In words, if B^T has a kernel then I may choose a $\tilde{p}$ in the kernel and then $B^T p = 0 \Rightarrow \max_v (v, B^T \tilde{p}) = 0$

An alternative (equivalent in finite dimension) is

2) $$\max_v \frac{(v, B^T p)}{\|v\|} \geq \beta \|p\| \quad \forall p.$$

or

3) $$\max_v \frac{(Bv, p)}{\|v\|} \geq \beta \|p\| \quad \forall p.$$

In infinite dimension, the choice of norm is crucial

The correct inf-sup condition for Stokes problem is

$$\sup_{v \in H_0^1} \frac{(\nabla \cdot v, p)}{\|v\|_1} \geq \beta \|p\|_0 \quad \forall p \in L_0^2$$

It is ~~called~~ called the <u>inf-sup</u> condition because it can also be written

$$\inf_{p \in L_0^2} \sup_{v \in H_0^1} \frac{(\nabla \cdot v, p)}{\|v\|_1} \geq \beta \|p\|_0$$

[We discussed error estimates last time, hence I will not go through that now]

We earlier discussed splitting / projection schemes for Navier-Stokes.

Here, we <u>discretized in time first</u> then in space. It enabled us to do certain tricks, that is; the equations were splitt into a series of simpler equations.

The IPCS was

1.) $$u^* - s(u^*) + \frac{\Delta t \nabla p^n}{\rho} = f^{n+1}$$

2.) $$-\nabla^2 \phi = -\frac{\rho}{\Delta t} \nabla \cdot u^*$$

3) $$u^{n+1} = u^* - \frac{\Delta t}{\rho} \nabla \phi$$

4) $$p^{n+1} = p^n + \phi$$

What is wrong with this scheme?

Boundary conditions!

Notice here that

- u^* has the right Dirichlet boundary conditions (I assume)

- Also φ is set up with homogenous Neumann conditions

$$\frac{\partial \varphi}{\partial n}$$

$\Rightarrow$ Some things are then correct.

$u^{n+1} \cdot n = u^{*} \cdot n$ since $\boxed{\frac{\Delta t}{\rho} \nabla \varphi \cdot n = 0}$

$u^{n+1} \cdot t \neq u^{*} \cdot t$ since $\frac{\Delta t}{\rho} \nabla \varphi \cdot t \neq 0$

in general.

Hence, $u^{n+1} \cdot t$ may be quite wrong

depending on $\nabla \varphi \cdot t$

Also, notice that

$$p^{n+1} = p^{n} + \varphi$$

If then is a time depedence

in p then $\frac{\partial p^{n+1}}{\partial n}$ may not equal $\frac{\partial p^{n}}{\partial n}$

I just illustrated an example of how things go wrong.

You can try to change e.g. the homogenous Neumann condition on φ to improve. I will argue that there is no general solution to this problem.

$\Rightarrow$ IPCS will always be first order.

Lifting

Very often, also in the material provided in this course, the homogenous Dirichlet problem is considered for error estimates.

The argument for this is that you can "always" reduce a non-homogenous problem to a homogenous.

Consider the problem

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = g \quad \text{on } \partial\Omega$$

there are many functions that equal g on the boundary. Let us pick one

$$\hat{u} = g \quad \text{on } \partial\Omega$$

but takes values in the whole of Ω.

Clearly then I can define $\bar{u}$ by

$$\bar{u} = u - \hat{u}$$

$\bar{u}$ satisfy the equation

$$-\Delta \bar{u} = -\Delta u - \Delta \hat{u} = f - \Delta \hat{u} \quad \text{in } \Omega$$

$$\bar{u} = u - \hat{u} = g - g = 0 \quad \text{on } \partial\Omega.$$

In other words, one may argue that it is sufficient to solve the problem

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

as any non-homogenous Dirichlet problem may be reduced to this.

For the most part of analysis you therefore see the space $H_0^1(\Omega)$ rather than $H_g^1(\Omega)$