Lecture 7

Stokes equations:

$$\begin{aligned} -\Delta u + \nabla p &= f \\ \nabla \cdot u &= 0 \end{aligned} \qquad + \text{ BC}$$

Weak formulation:

$$\int_\Omega (-\Delta u + \nabla p = f) \cdot v \, dx$$

$$\hookrightarrow \int_\Omega (-\Delta u + \nabla p)\, v \, dx = \int_\Omega f v \, dx$$

$$\|$$

$$\int_\Omega (\nabla u : \nabla v - p \, \nabla \cdot v) \, dx$$

$$- \int_{\partial\Omega} \left(\frac{\partial u}{\partial n} - p \cdot n\right) v \, ds$$

2)

$v = 0$ at $\partial\Omega_D$

Here,

$$-\int_{\partial\Omega} \left(\frac{\partial u}{\partial n} + p \cdot n\right) v = -\int_{\partial\Omega_D} \left(\frac{\partial u}{\partial n} + p \cdot n\right) v \, ds$$

$$+ \int_{\partial\Omega_N} \underbrace{\left(\frac{\partial u}{\partial n} - p \cdot n\right)}_{g_N} v \, ds$$

$$= -\int g_N \cdot v \, ds$$

minus sign changes to +

when moving from left hand-side

to right hand-side.

As such we end up with

Find u, p such that

$$\int_\Omega \nabla u : \nabla v - \int_\Omega p \nabla \cdot v = \int_\Omega f v + \int_{\partial\Omega_N} g_N v$$

$$\int_\Omega \nabla \cdot u \, q = 0$$

$$\forall \; u, q$$

We write this as a finite element problem:

Find u_h, p_h where

$$u_h = \sum_i u_i N_i \quad , p_h = \sum p_j L_j$$

such that

$$\int_\Omega \nabla u_h : \nabla N_k - \int_\Omega p_h \nabla \cdot N_k = \int_\Omega f N_k + \int_{\partial\Omega} g_N N_k$$

$$\int_\Omega \nabla \cdot u_h L_\ell = 0$$

This can be written as

$$A u_h + B^T p_h = f_h$$

$$B u_h = 0$$

where

$$A_{ij} = \int_\Omega \nabla N_i : \nabla N_j \, dx$$

$$B_{il} = \int_\Omega \nabla \cdot N_i \, L_l \, dx$$