Week 6

Navier-Stokes

$$\frac{\partial v}{\partial t} + \boxed{v \cdot \nabla v} = -\frac{1}{\rho} \nabla p \boxed{+ \nu \nabla^2 v} + g$$

$$\nabla \cdot v = 0$$

On non-dimensionalized form

$$\frac{\partial v^*}{\partial t} + \boxed{\left(v^* \cdot \nabla^* v^* \right)} = -\nabla^* p^* \boxed{+ \frac{1}{Re} \nabla^{*2} u^*} + f^*$$

$$Re = \frac{V L}{\nu}$$

Our example is

$$-\mu \Delta u + v \cdot \nabla u = f$$

diffusivity

velocity

viscosity if u is a vector

<u>Pechlet number</u>

$$= \frac{\text{advective transport rate}}{\text{diffusive transport rate}}$$

$$= \frac{Lv}{\mu}$$

3)

Reynolds number
or Pechlet number
is a kind of a
condition number,
that is it measures
the conditioning of
the operator relative
to viscous / diffusive
transport rate

Last time we saw that

$$a(u,v) = \int_\Omega \mu \nabla u \cdot \nabla w + \int v \cdot \nabla u \, w$$

$$\leq (\mu + \|v\|_\infty C_\Omega) \|u\|_1 \|v\|_1$$

The next question is then (in linear algebra terms)

$$x^T A x \geq \alpha \|x\|^2$$

And this is kind of cool!

Assume that $\nabla \cdot v = 0$ (which in most cases is ok)

Let $c_v(u,w) = \int v \cdot \nabla u \, w$

We can then show that

$$C_v(u, w) = -C_v(w, u)$$

which means that $(w = u)$

$$C_v(u, u) = -C_v(u, u) = 0$$

linear algebra equivalent

$$x^T C x = 0 \quad \forall x$$

C is anti symmetric,

i.e. $C = -C^T$.

Example

$$C = \frac{1}{h}\begin{bmatrix} 0 & -1 & & & & \\ 1 & 0 & -1 & & & \\ & 1 & 0 & -1 & & \\ & & 1 & 0 & -1 & \\ & & & 1 & 0 & -1 \\ & & & & 1 & 0 \end{bmatrix}$$

2-order central difference

$$\frac{u_{i+1} - u_{i-1}}{2h}$$

$\Rightarrow$ Transport is typically anti-symmetric $\rightarrow$ (energy preservation.)

We wanted to show that

$$c_v(u,w) = -c_v(w,u)$$

$$\|$$

$$\int_\Omega v\cdot\nabla u\, w \underset{\text{integration by parts}}{=} -\int_\Omega v\cdot\nabla w\, u \overbrace{-\int_\Omega \nabla\cdot v\, u\, w}^{\nabla\cdot v = 0} \underbrace{+\int_\Gamma u\, w\, v\cdot n}_{\text{Boundary conditions}}$$

$$\boxed{= -\int_\Omega v\cdot\nabla w\, u = -c_v(w,u)}$$

The conclusion is that

$$\alpha = \mu$$

$$C = (\mu + \|v\|_\infty C_\Omega)$$

The error estimate becomes:

$$|u - u_n|_1 \leq \frac{C}{\alpha} \| \quad \| \leq \frac{(\mu + \|v\|_\infty C_\Omega)}{\mu}$$

we saw several different estimates here earlier

$C_\Omega = L$

~~$\|v\|_\infty$ could be max or mi~~

If μ is small then this is similar to Re.

Let us now consider

Stokes problem

$$-\nu \Delta u - \nabla p = f$$

$$\nabla \cdot u = 0$$

We start with ~~an~~ a linear algebra type approach.

I will derive weak forms etc next time.

We saw that for the convection diffusion problem the approximation was tied to the propertives

$$x^T A x > 0 \quad \forall x \qquad \text{and} \qquad x^T A y < \infty \quad \forall x, y$$

More precisely we should ~~do~~ perhaps say

$$x^T A x \geq \alpha \|x\| \quad \forall x \qquad \text{and} \qquad x^T A y \leq C \|x\| \|y\| \quad \forall x, y$$

and we saw that the choice of norm $\| \cdot \|$ was crucial.

The Stokes problem is of the form

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$

When is this problem solvable, i.e when does it have a unique solution (with x, y bounded by c, d)

For Stokes problem, as will be detailed next time, $A \in \mathbb{R}^{n \times n}$ and is invertable, ~~the~~ while $B \in \mathbb{R}^{m \times n}$ with $m \ll n$.

Let us use that A is invertable.

$$Ax + B^T y = c \qquad 1)$$

$$Bx = d \qquad 2)$$

Multiply 1) with A^{-1}

$\Rightarrow$

$$x = A^{-1}(c - B^T y) \qquad 3)$$

insert 3) into 2)

$\Rightarrow$

$$BA^{-1}(c - B^T y) = d$$

or $\boxed{-BA^{-1}B^T y = d - BA^{-1}c}$

$BA^{-1}B^T$ is called the <u>Schur complement</u>.

Clearly the system

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$

is solvable if

$$\boxed{-BA^{-1}B^T y = d - BA^{-1}c}$$

is solvable

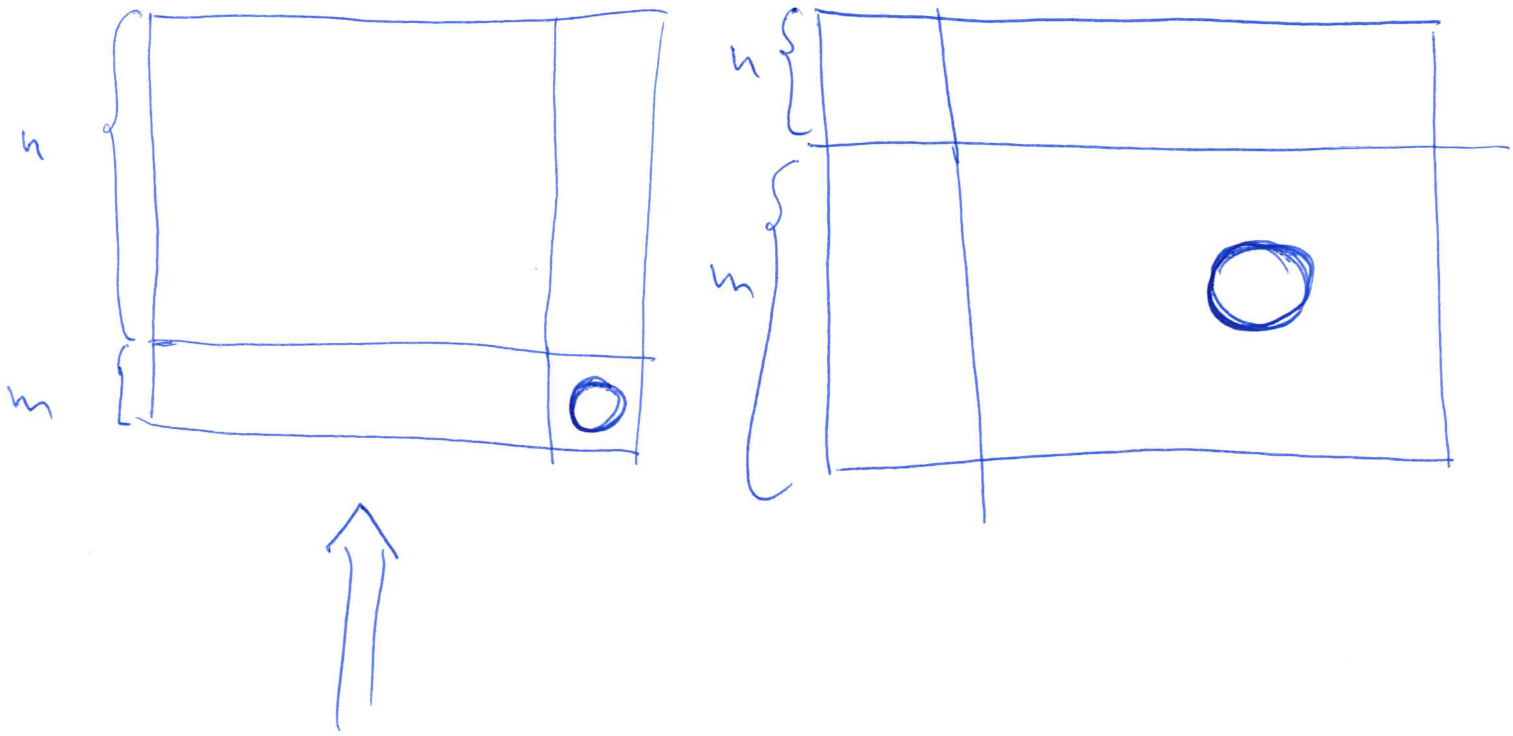
$$\boxed{\text{and} \quad x = A^{-1}(c - B^T y)}$$

when is

$BA^{-1}B$ invertible ?

A is $\mathbb{R}^{n\times n}$

and B is $\mathbb{R}^{m\times n}$.



We are in this

situation $n \gg m$.

What are appropriate conditions on

B ?