Lecture 5

Last time I listed

the following abstract properties

$$a(u,u) \geq \alpha \|u\|_1^2 \qquad \forall\, u \in H^1$$

$$a(u,v) \leq C \|u\|_1 \|v\|_1$$

when

$$\|u\|_1 = \sqrt{\int_\Omega \left(u^2 + (\nabla u)^2\right) dx}$$

I said that I

would relate α, C to

Reynolds number

I also said

that

$a(u,u) \geq \alpha \|u\|_1^2$ was "like" $x^T A x > 0$

and that

$a(u,v) \leq C \|u\|_1 \|v\|_1$ was "like" $x^T A x < \infty$

Let us the consider

$$u_{xx} = f \quad \text{on} \quad (0,1)$$

~~$u = 0$ on~~

$$u(0) = u(1) = 0.$$

We know that in this case we can express the solution as a Fourier series

$$u = \sum_{k=1}^{\infty} u_k \sin(k\pi x) \qquad (*)$$

when $u_k = \frac{1}{(k\pi)^2} \int f \sin(k\pi x)$

Let us assume $(*)$ and check if

$$x^T A^{\cancel{*}} x > 0$$

and

$$x^T A x < \infty$$

Here

$$a(u,u) = \int_\Omega -\nabla u \cdot \nabla v = \int_\Omega -\Delta u \cdot v$$

$$= \int \left((k\pi)^2 \left(\sum_{k=1}^{\infty} u_k \sin(k\pi x) \right) \right) \left(\sum_{k=1}^{\infty} u_k \sin(k\pi x) \right)$$

$$= \int (k\pi)^2 \sum_{k=1}^{\infty} u_k^2 \sin^2(k\pi x)$$

$$\boxed{\begin{array}{c} \sin(k\pi x) \\ \perp \\ \sin(l\pi x) \\ \text{if} \\ k \neq l \end{array}}$$

We can conclude that

$$a(u,u) > 0$$

However

$$a(u,u) \xrightarrow[k \to \infty]{} \infty$$

In other words

$x^T A \cancel{x} < \infty$ is not true!

However,

our condition is slightly different. We want $a(\cdot,\cdot)$ to be bounded relative to the $\|\cdot\|_1$ norm, i.e.,

$$a(u,v) \leq C\|u\|_1\|v\|_1$$

Let us re-do the exercise to check the abstract conditions.

Consider

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Variational form

1. multiply by test function
2. integrate (by parts)

$$\int_\Omega -\Delta u \cdot v \, dx = \int_\Omega f \cdot v \, dx$$

$\Downarrow$ integration by parts

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f \cdot v \, dx$$

Hence, the ~~abstra~~ variational / weak problem is

Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = L(v) \qquad \forall v \in H_0^1(\Omega)$$

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad L(v) = \int_\Omega f v \, dx$$

The abstract condition $a(u,v) \leq C \|u\|_1 \|v\|_1$ can be shown as follows:

$$a(u,v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx$$

$$\leq \left(\int_{\Omega} (\nabla u)^2 \right)^{1/2} \left(\int (\nabla v)^2 \right)^{1/2}$$

Cauchy schwartz

$$\leq \left(\int (u^2 + (\nabla u)^2) \right)^{1/2} \left(\int v^2 + (\nabla v)^2 \right)^{1/2}$$

$(\nabla u)^2 \leq (u)^2 + (\nabla u)^2$

$$\leq \underline{1} \, \|u\|_1 \|v\|_1$$

$\boxed{\text{Poincaré's inequality}}$

For all functions in $H_0^1(\Omega)$ we have

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}$$

The result follows directly from Taylor series:

$$f(y) \simeq f(x) + hf'(\bar{x})$$

In $H_0^1(\Omega)$ we have that ~~[illegible]~~ the function is zero at the boundary. Let then x be at the boundary ~~then~~ and y ~~[illegible]~~ is inside. Then

$$|f(y)| < |hf'(\bar{x})|$$

Due to Poincare, we get 10)

$$a(u,u) = \int (\nabla u)^2$$ splitt in two

$$\geq \frac{1}{2} \int (\nabla u)^2 + \frac{1}{C^2} u^2$$

$$\geq \frac{1}{2} \min\left(1, \frac{1}{C^2}\right) \int \nabla u^2 + u^2$$

$$= \frac{1}{2} \min\left(1, \frac{1}{C^2}\right) \|u\|_1^2$$

We have now done a lot of abstract math.

$\boxed{\text{How does this relate to the Reynolds number?}}$

Consider then the convection diffusion ~~the~~ problem

$$-\mu \Delta u + w \cdot \nabla u = f \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \Omega$$

μ viscosity

w fluid velocity.

Reynolds number relate

μ, w and ~~the~~ length

of Ω.

Weak form of the problem

Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = L(v) \quad \forall\, v \in H_0^1(\Omega)$$

Here

$$a(u,v) = \int_\Omega (\mu \nabla u \cdot \nabla v + w \cdot \nabla u \cdot v)\, dx$$

$$L(v) = \int_\Omega f\, v$$

You check !

$$a(u,v) = \int_\Omega \nu \nabla u \cdot \nabla v + \int_\Omega w \nabla u \cdot v$$

$$\leq \nu \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} +$$

$$+ \|w\|_{L^\infty} \|\nabla u\|_{L^2} \cdot \|v\|_{L^2}$$

$$\leq (\nu + \|w\|_{L^\infty}) \left(\|\nabla u\|_{L^2} \|\nabla v\|_{L^2} + \|\nabla u\|_{L^2} \cdot \|v\|_{L^2} \right)$$

$$\leq (\nu + \|w\|_{L^2}) \left(\|u\|_{H^1} \|v\|_{H^1} \right)$$