The strange world of PDEs and their discretizations.

Consider the problem

$$Ax = b$$

$A \in \mathbb{R}^{n,n}$ is positive definite and symmetric

$\Longrightarrow$ unik, stable solution.

We have

$$x = A^{-1} b$$

$$\| x \| = \| A^{-1} b \| \leq \| A^{-1} \| \, \| b \|$$

and

$$\| Ax \| = \| b \| \, .$$

For PDEs we do not have the analog of

$$\|Ax\| \leq \|b\|$$

or

$$\|x\| \leq \|A^{-1}\| b.$$

Instead, we have something like

$$\|A^{1/2} x\| \leq \|A^{-1/2} b\|$$

Remark for A

positive def. and symmetric

we may define A^s

for any $s \in \mathbb{R}$

simply in terms of

the eigenvalue decomposition

Let $A = U \Lambda V$

where Λ are the

eigenvalues.

Then $A^s = U \Lambda^s V$

A^s may not be

unique.

Let

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$ ($-\Delta$ on a mesh with two elements)

Let

$$B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$ (gradient on a mesh with two elements)

Then $B^T B = A$

Even thoug $A \in \mathbb{R}^{2\times 2}$, $B \in^{2\times 3}$

Hence, for

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$Ax = b$$

We could hope for something like

$$\| Bx \| \leq \| B^{-T} b \|$$

since

$$B^T B x = Ax = b$$ multiply by B^{-T}

$$\| Bx \| = \| B^{-T} b \|$$

However,

B^{-T} is "difficult" to define since $B \in \mathbb{R}^{2\times 3}$.

Let us now consider a PDE.

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \qquad (1)$$

We know that formally (1) is not well defined ~~for~~ ~~as~~ as (1) is its strong formulation.

We are not allowed to ~~say~~ do

$$u = -\Delta^{-1} f$$

and $\|u\| \leq \|-\Delta^{-1} f\|$

What we have is the weak form: Find $u \in H_0^1(\Omega)$

$$\int_\Omega \nabla u : \nabla v = \int_\Omega f\, v \quad \forall\, v \in H_0^1(\Omega)$$

From the weak form

I would ~~to~~ like to

write

$$\|\nabla u\| \leq \|(\nabla)^{-1} f\|$$

Problems here again

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The correct thing to write here is

$$\|u\|_1 \leq \|f\|_{-1}$$

should be semi-norms.

Here

$$\|u\|_1^2 = \int (\nabla u)^2$$

what about $\|f\|_{-1}$?

Lets play!

Say $-\Delta u = f$

$-\Delta v = g$

solutions defined weakly

We have

$$(u,v)_1 = (-\Delta u, v) = (\nabla u, \nabla v) = (u, -\Delta v)$$

Analogously, lets say that

$$(f,g)_{-1} = (-\Delta^{-1} f, g) = (f, -\Delta^{-1} g)$$

Now

$$(u,v)_1 = (-\Delta u, v) = (f, v)$$

$$\underset{\substack{\uparrow \\ v = -\Delta^{-1} g}}{=} (f, -\Delta^{-1} g) = (f,g)_{-1}$$

Let $u = \sin(k\pi x)$

Then

$$(u,u)_1 = (-\Delta u, u)$$

$$= (\pi k)^2 (u,u) = (\pi k)^2 \|u\|_0^2$$

Further

$$(u,u)_{-1} = ((-\Delta)^{-1} u, u) = \frac{1}{(\pi k)^2} (u,u)$$

$$= \frac{1}{(\pi k)^2} \|u\|_0^2$$

High frequencies are important in H^1, with weight $(k\pi)^2$

but inversly less important in H^{-1}

H^{-1} is the dual space

of H_0^1.

The formal, but less instructive

definition is that

$$\|f\|_{-1} = \sup_{v \in H_0^1} \frac{(f, v)}{\|v\|_1}$$

There is a distinction between functions and functionals.

A functional is something that takes a function and produce a number.

Any function can together with the L^2 inner product be turned into a functional!!

Let f be some function.

Then the corresponding functional is when applied to v

$$f(v) = \int_{\Omega} f \, v \; dx.$$

Hence, in some sense

the ~~distingz~~ distinction

between functions and

functionals are not that

important in practice.

But functionals may be

defined for "generalized functions"

such as Dirac's delta.