

The strange world of

1)

PDEs and their discretizations.

Consider the problem

$$Ax = b$$

$A \in \mathbb{R}^{n,n}$ is positive definite and symmetric

\Rightarrow unique, stable solution.

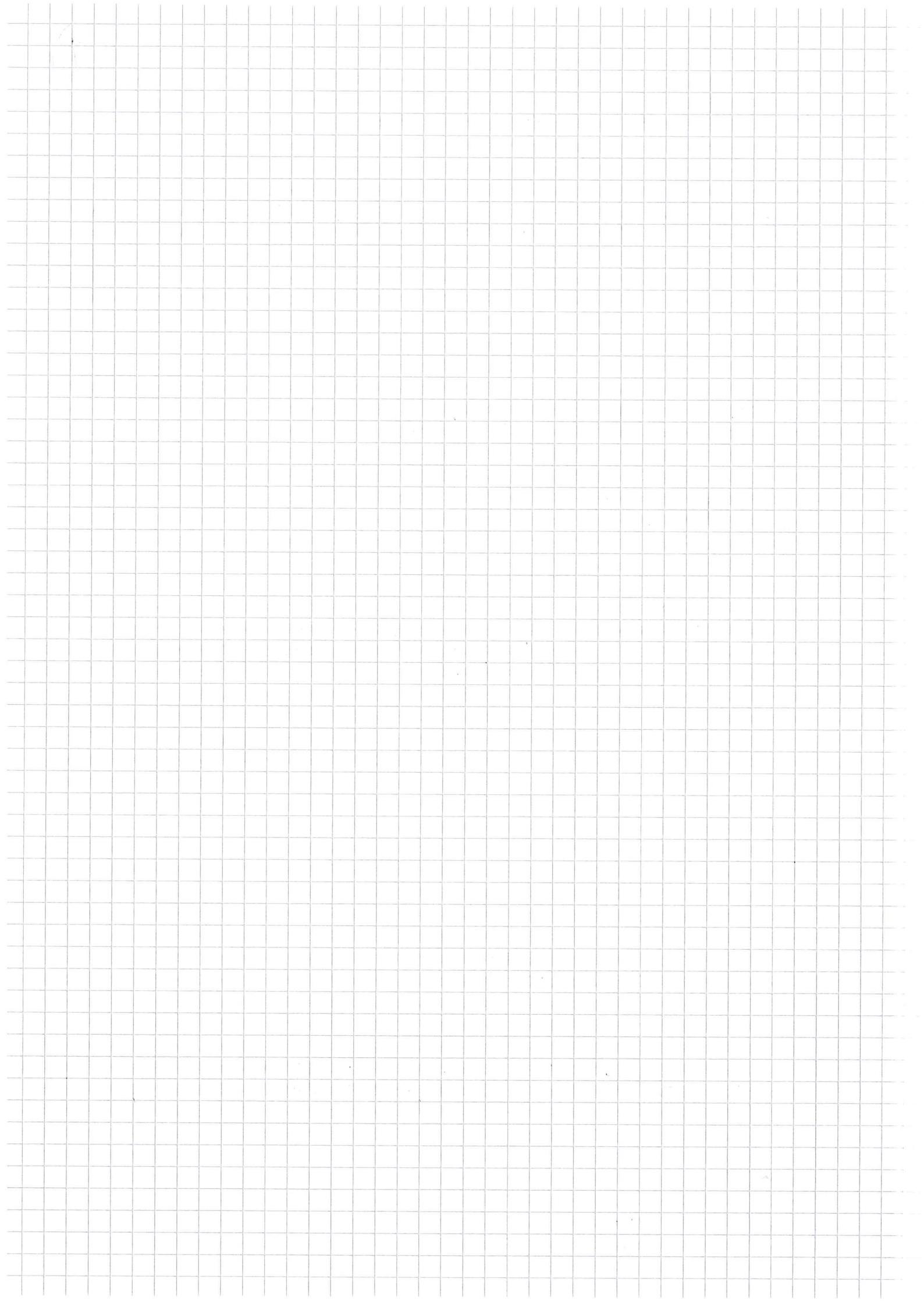
We have

$$x = A^{-1}b$$

$$\|x\| = \|A^{-1}b\| \leq \|A^{-1}\| \|b\|$$

and

$$\|Ax\| = \|b\| .$$



For PDEs we do not

2)

have the analog of

$$\|Ax\| \leq \|b\|$$

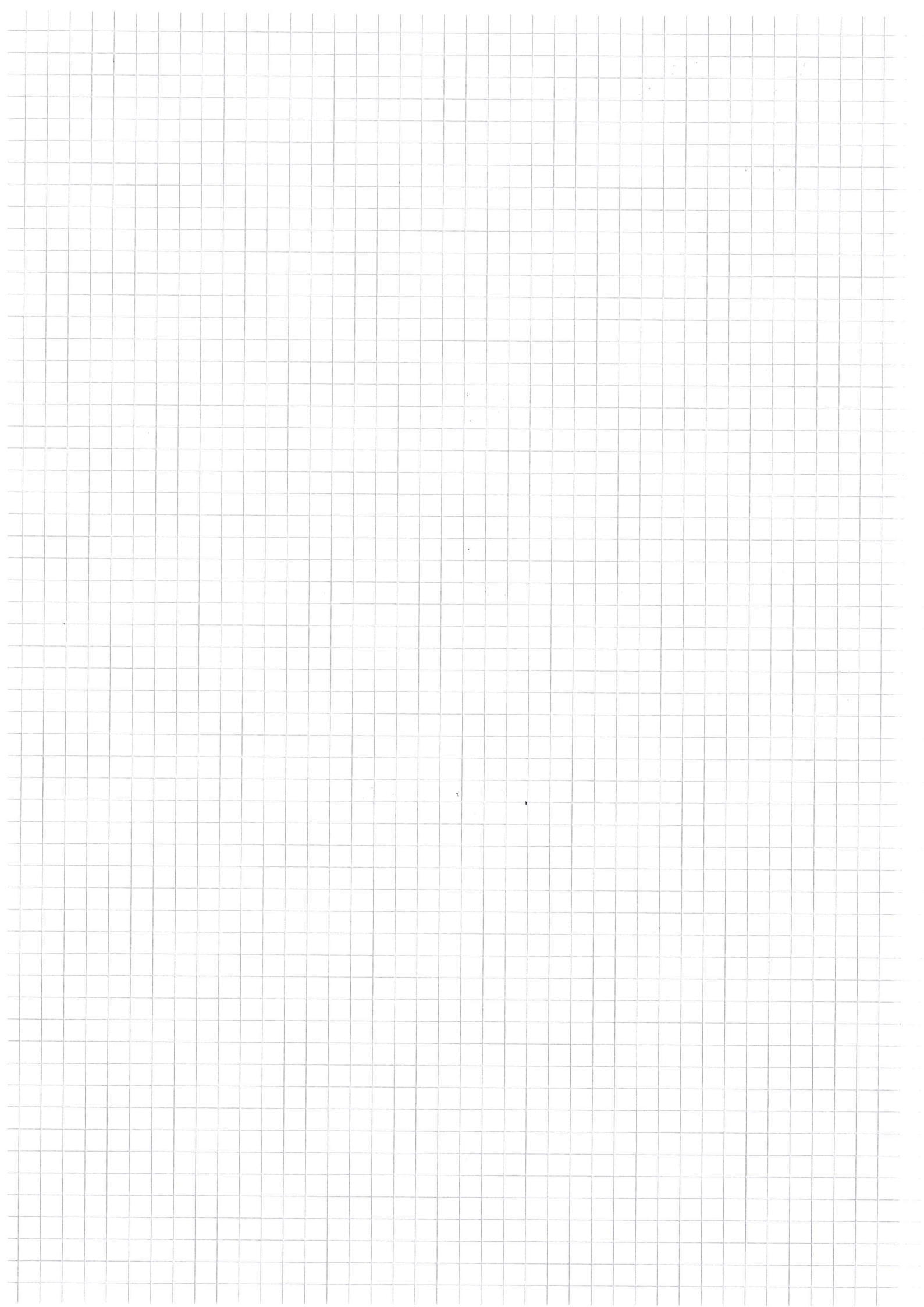
or

$$\|x\| \leq \|A^{-1}\| \|b\|$$

Instead, we have something

like

$$\|A^{1/2} x\| \leq \|A^{-1/2} b\|$$



Remark for A

3)

positive det. and symmetric

we may define A^s

for any $s \in \mathbb{R}$

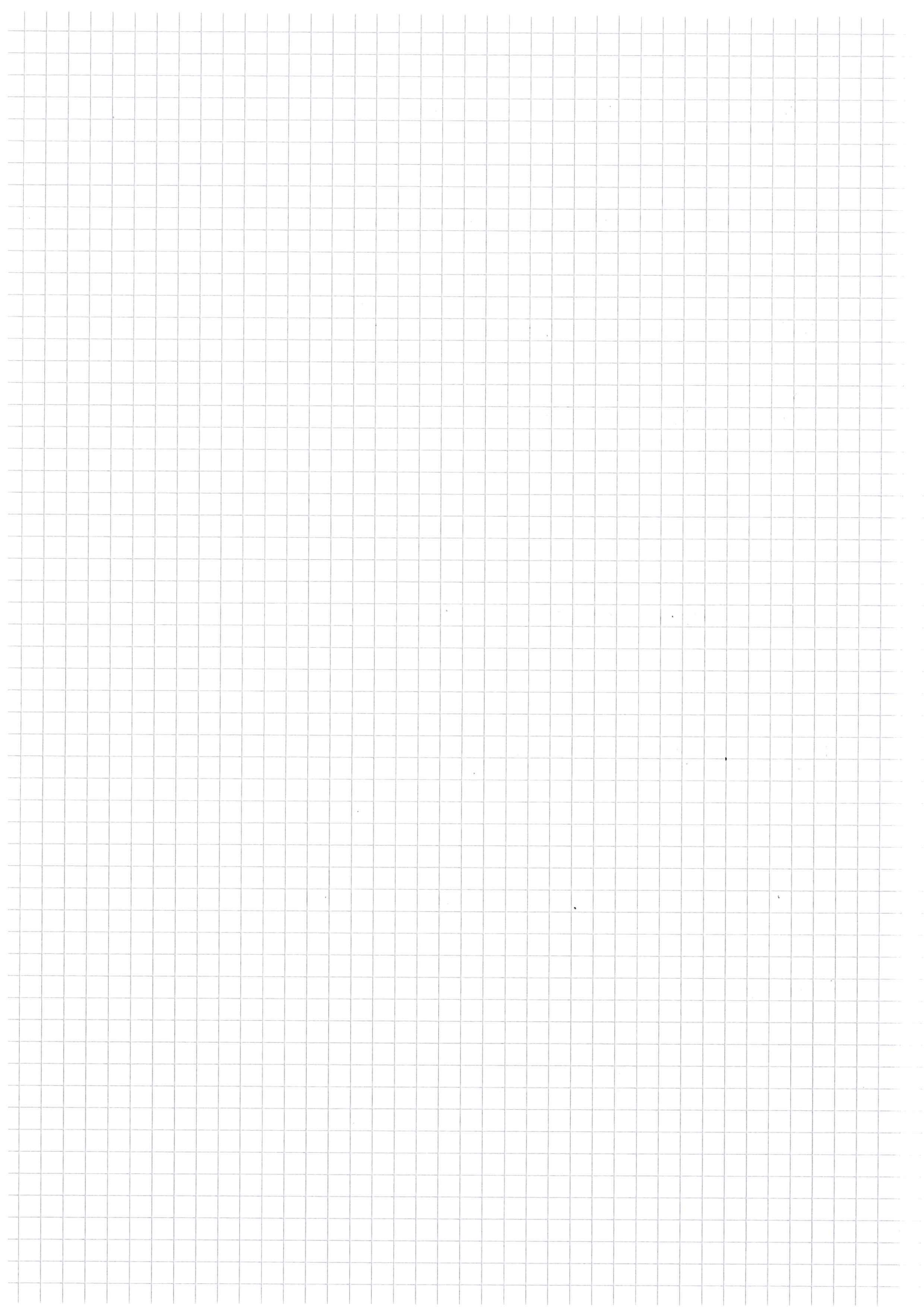
simply a in terms of

the eigenvalue decomposition

$$\text{Let } A = U \Lambda V$$

where Λ are the
eigenvalues.

$$\text{Then } A^s = U \Lambda^s V$$



4)

A^s may not be
unique.

Let

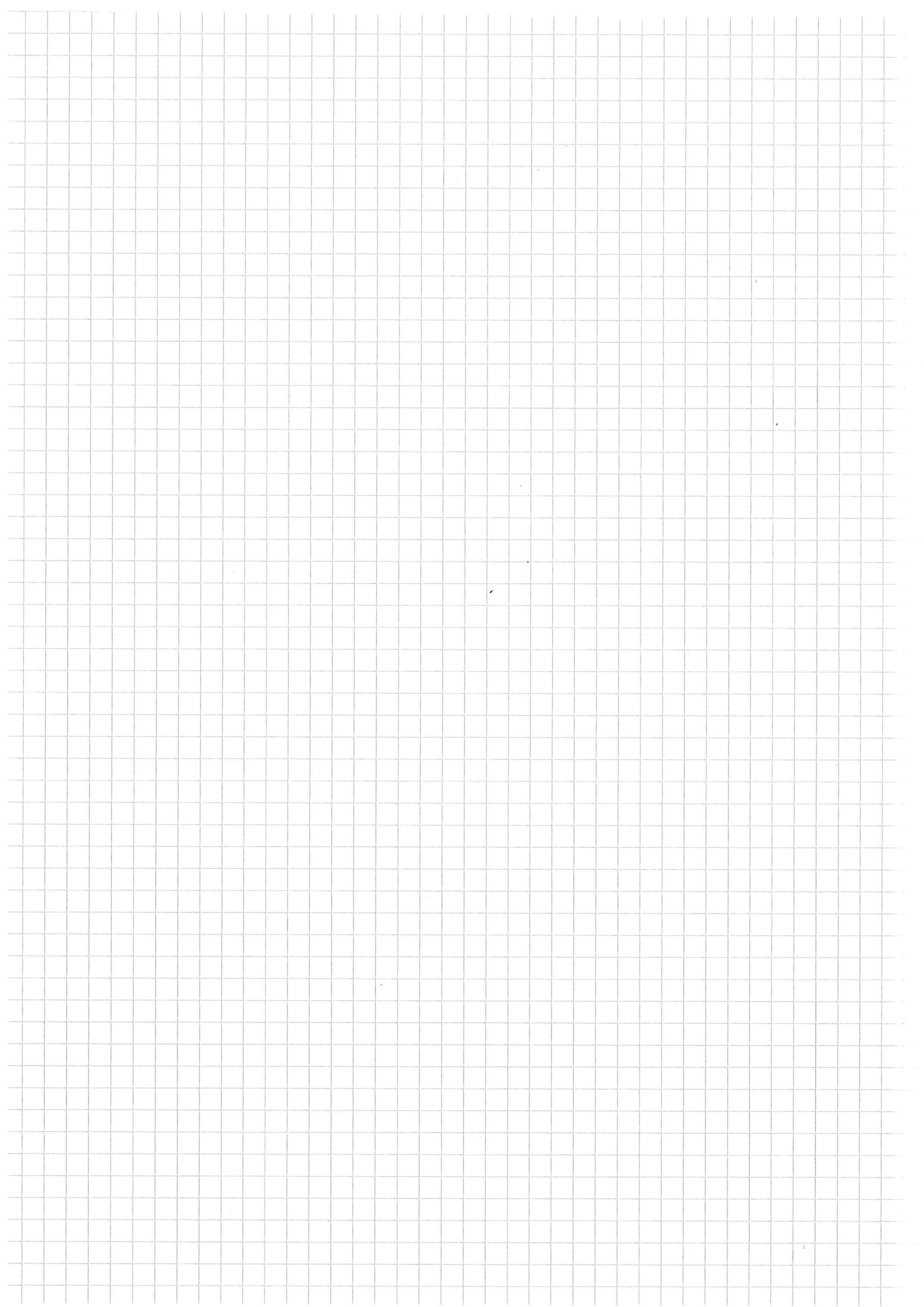
$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \left(\begin{array}{l} -\Delta \text{ on} \\ \text{a mesh} \\ \text{with two elements} \end{array} \right)$$

Let

$$B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \quad \left(\begin{array}{l} \text{gradient on} \\ \text{a mesh with} \\ \text{two elements} \end{array} \right)$$

Then $B^T B = A$

Even though $A \in \mathbb{R}^{2 \times 2}$, $B \in \mathbb{R}^{2 \times 3}$.



Hence, for

5

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$Ax = b$$

We could hope for something

like

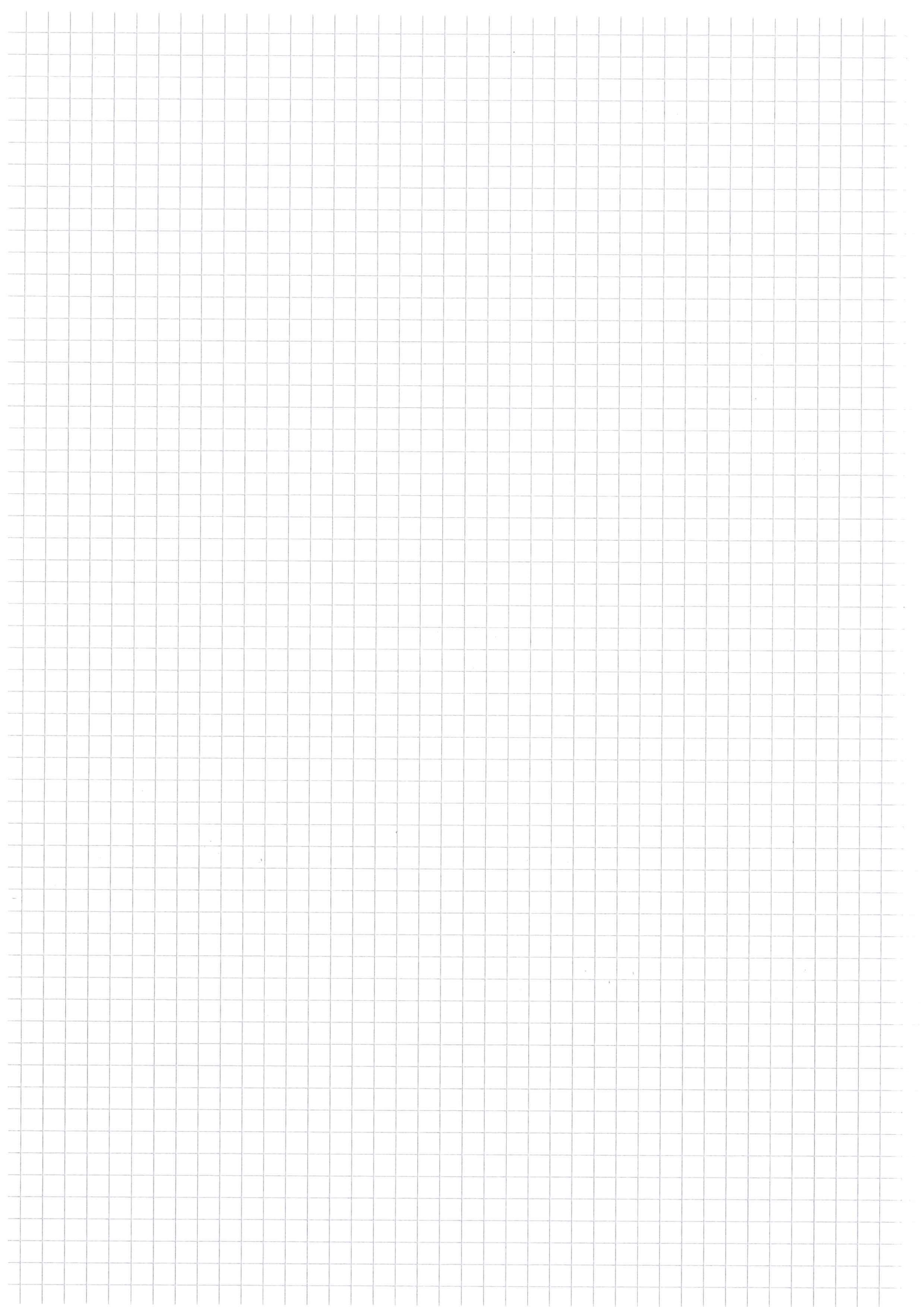
$$\|Bx\| \leq \|B^{-T}b\|$$

since

$$B^T Bx = Ax = b$$

multiply by B^{-T}

$$\|Bx\| = \|B^{-T}b\|$$



However,

6)

B^{-T} is "difficult" to
define since $B \in \mathbb{R}^{2 \times 3}$

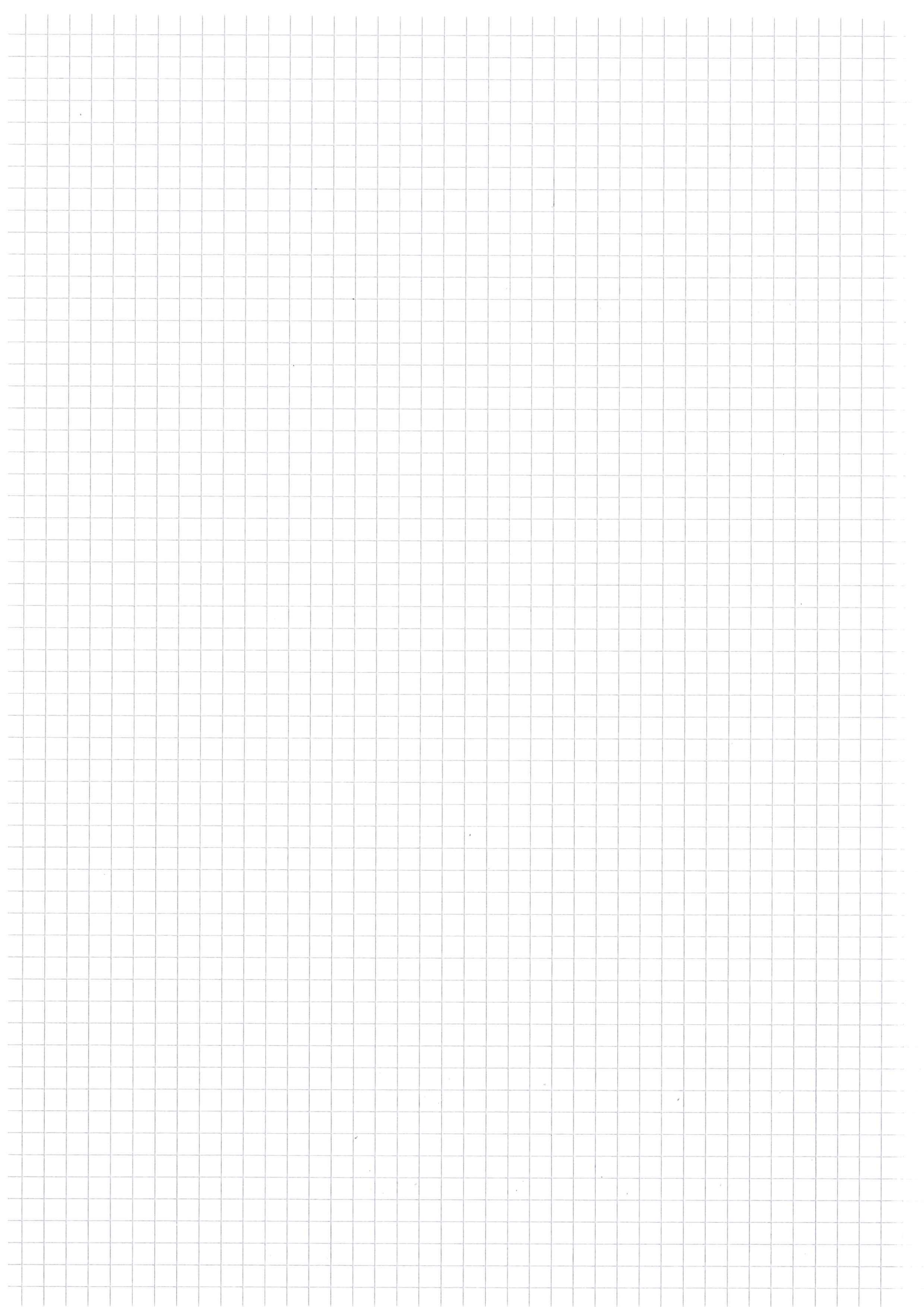
Let us now consider a
PDE.

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (1)$$

We know that formally

(1) is not well defined ~~as~~

as (1) is its strong formulation.



7)

We are not allowed to
~~say~~ do

$$u = -\Delta^{-1} f$$

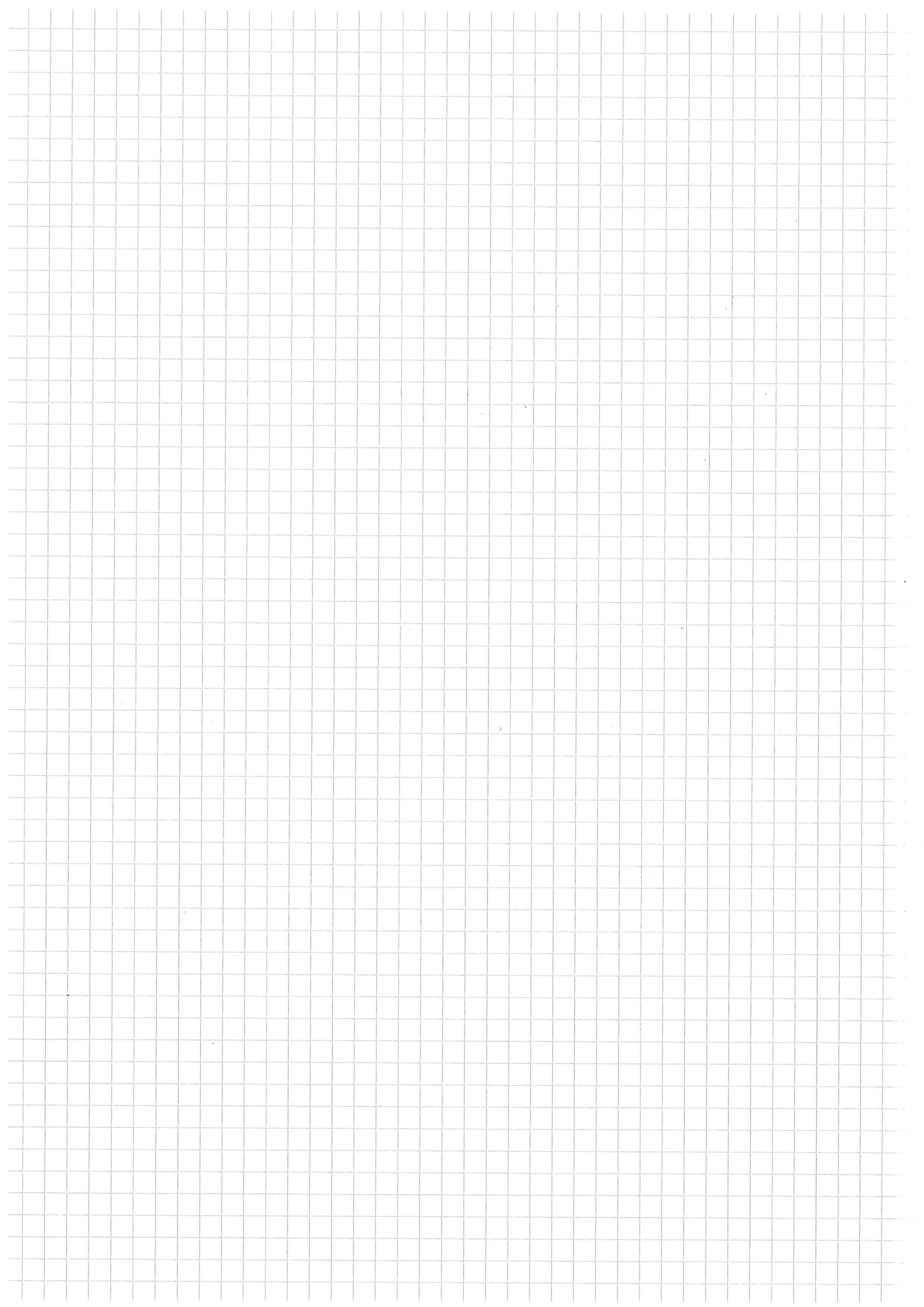
and $\|u\| \leq \|-\Delta^{-1} f\|$

What we have is the

weak form : Find $u \in H_0^1(\Omega)$

$$\int_{\Omega} \nabla u : \nabla v = \int_{\Omega} f v \quad \forall$$

$$v \in H_0^1(\Omega)$$



8)

From the weak form

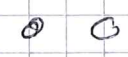
I would ~~be~~ like to

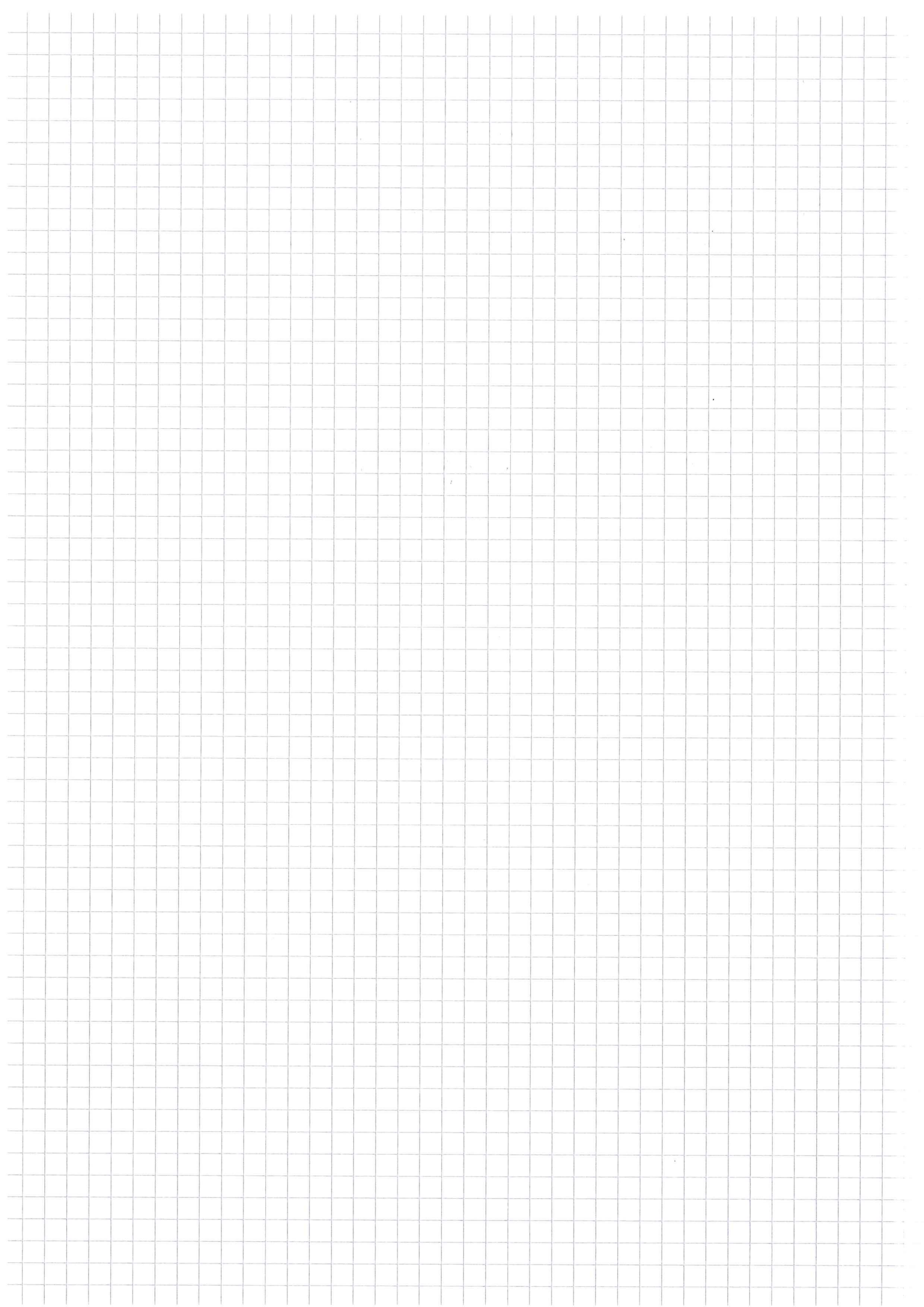
write

$$\| \nabla u \| \leq \| (\nabla)^{-1} f \|$$



Problems here
again





9)

The correct thing

to write here is

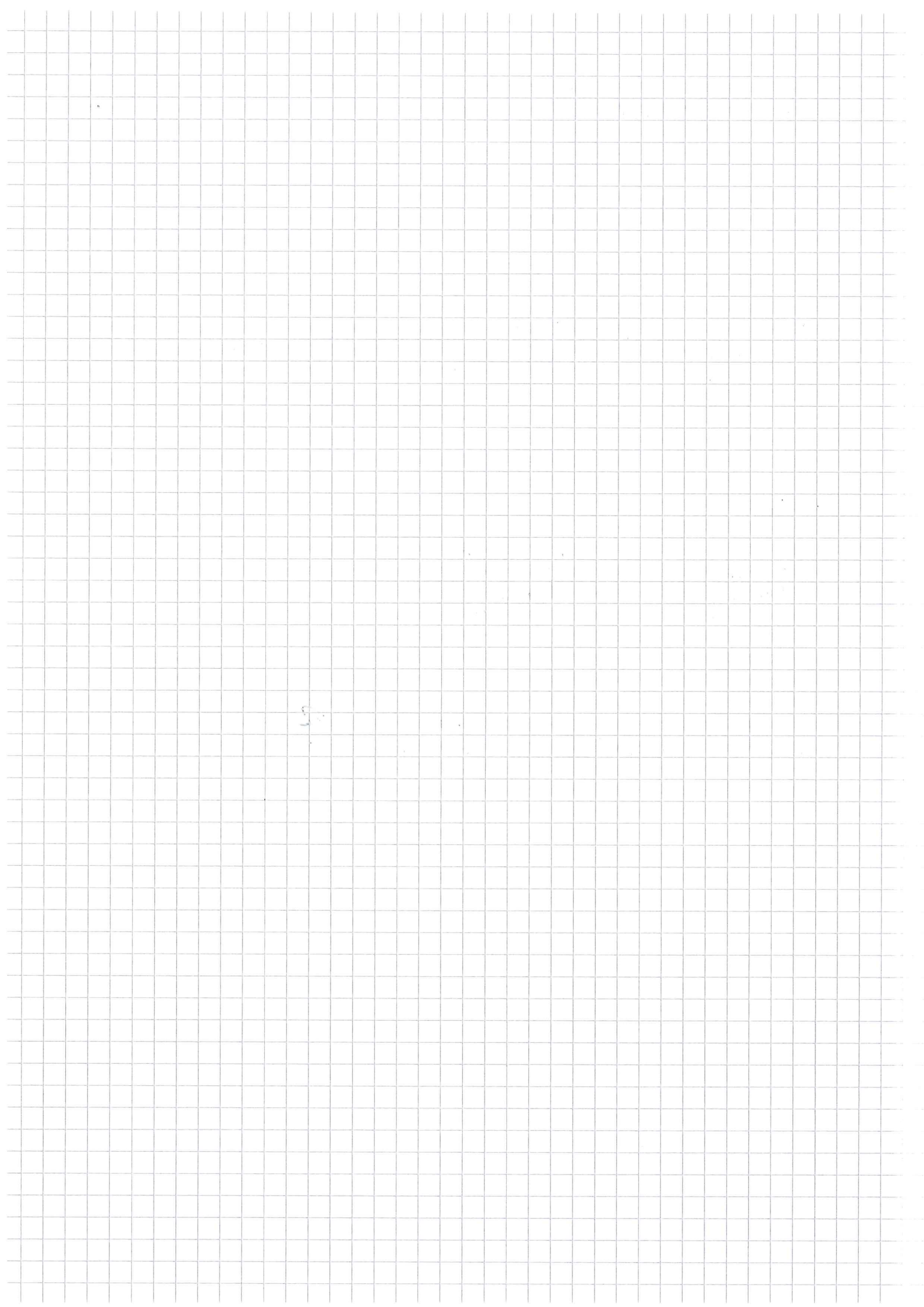
$$\|u\|_1 \leq \|f\|_{-1}$$

← should
be semi-norms.

here

$$\|u\|_1^2 = \int (\nabla u)^2$$

what about $\|f\|_{-1}$?



Lets play !

Say

$$\begin{aligned}
 -\Delta u &= f \\
 -\Delta v &= g
 \end{aligned}$$

← solutions defined weakly

We have

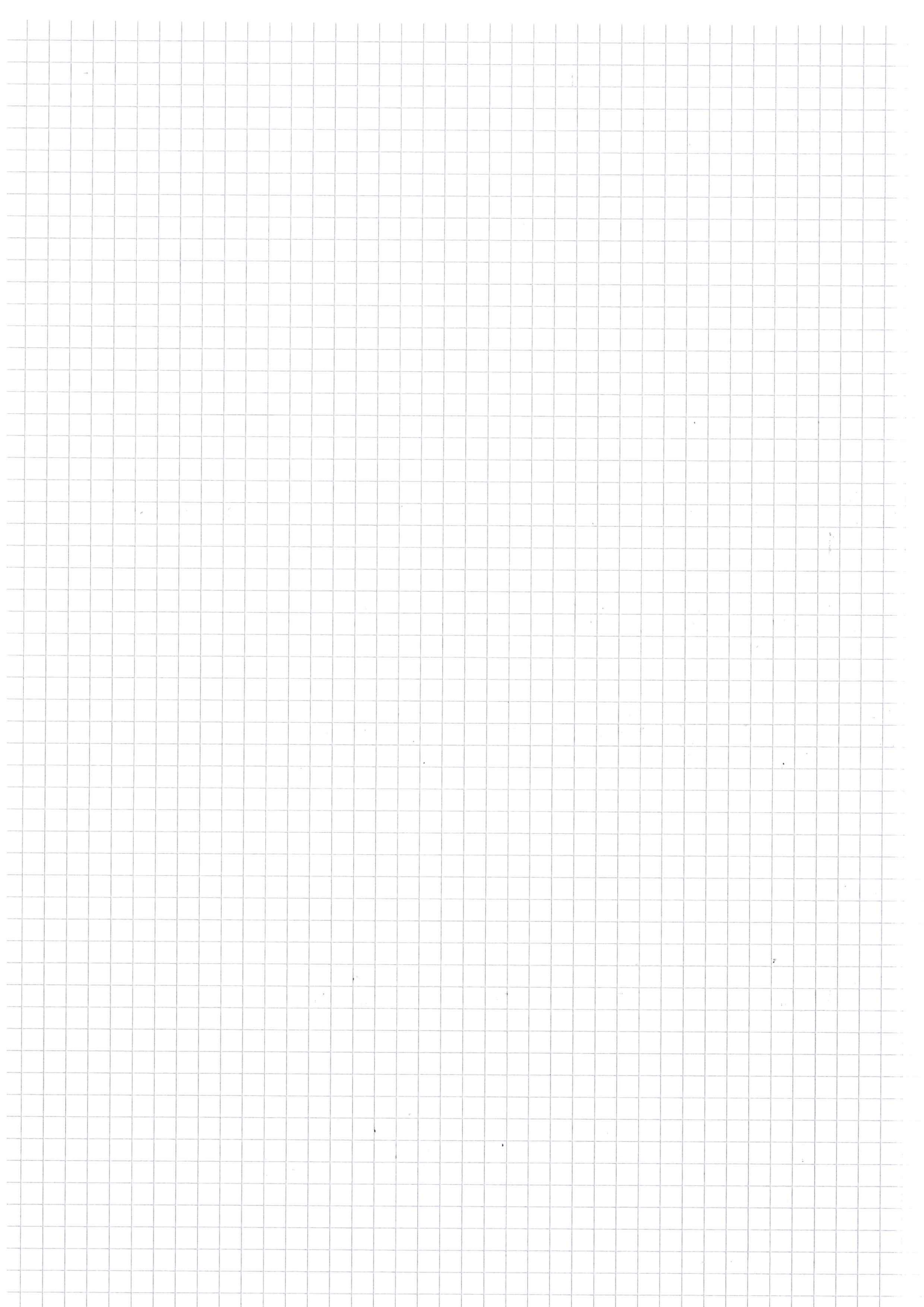
$$(u, v)_1 = (-\Delta u, v) = (\nabla u, \nabla v) = (u, -\Delta v)$$

Analogously, lets say that

$$(f, g)_{-1} = (-\Delta^{-1} f, g) = (f, -\Delta^{-1} g)$$

Now

$$\begin{aligned}
 (u, v)_1 &= (-\Delta u, v) = (f, v) \\
 &= (f, -\Delta^{-1} g) = (f, g)_{-1} \\
 \uparrow \\
 v &= -\Delta^{-1} g
 \end{aligned}$$



$$\text{Let } u = \sin(k\pi x)$$

71)

Then

$$\begin{aligned}(u, u)_1 &= (-\Delta u, u) \\ &= (\pi k)^2 (u, u) = (\pi k)^2 \|u\|_0^2\end{aligned}$$

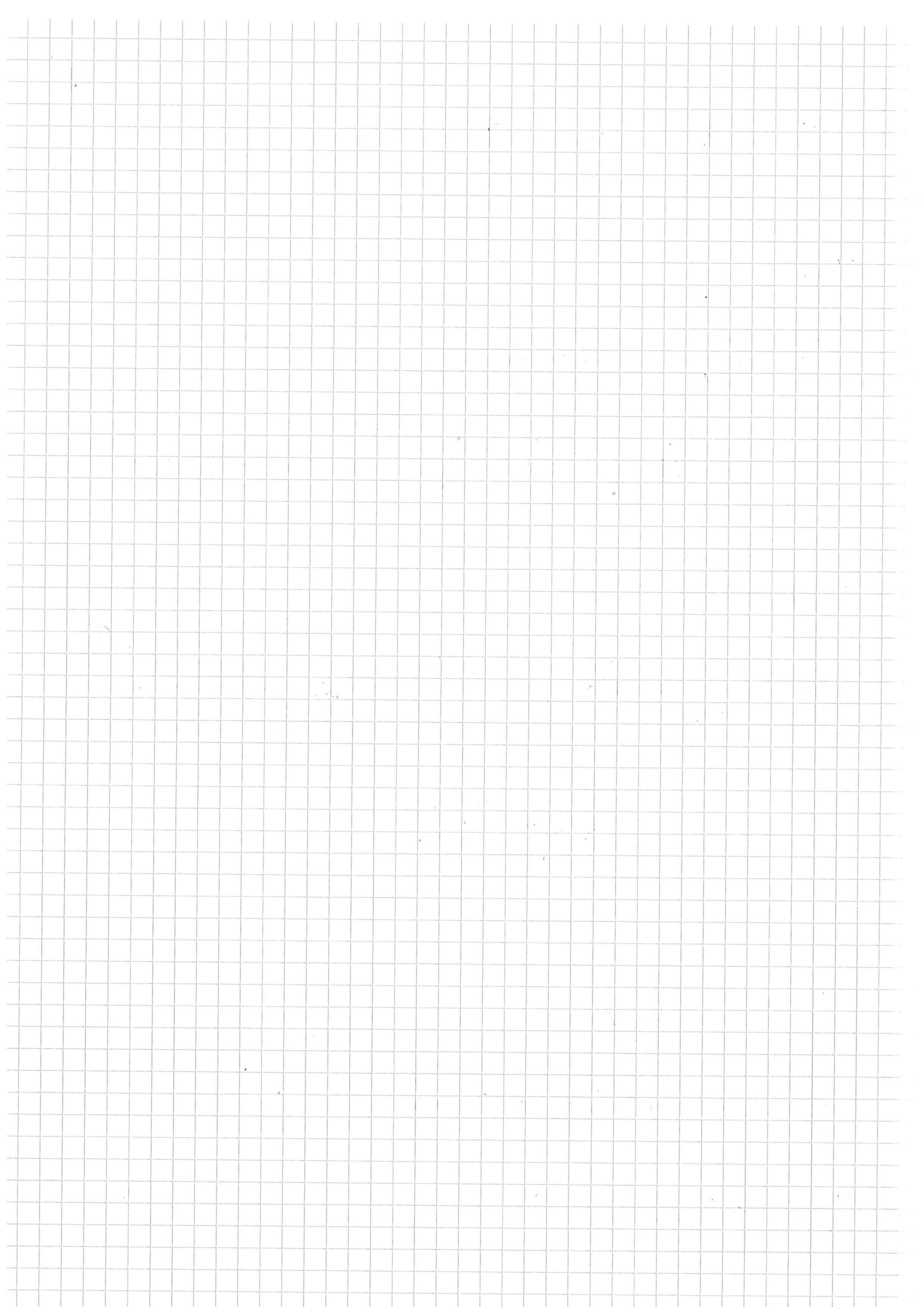
Further

$$\begin{aligned}(u, u)_{-1} &= ((-\Delta)^{-1} u, u) = \frac{1}{(\pi k)^2} (u, u) \\ &= \frac{1}{(\pi k)^2} \|u\|_0^2\end{aligned}$$

High frequencies are important

in H^1 , with weight $(k\pi)^2$

but ~~is~~ inversely less important in H^{-1}

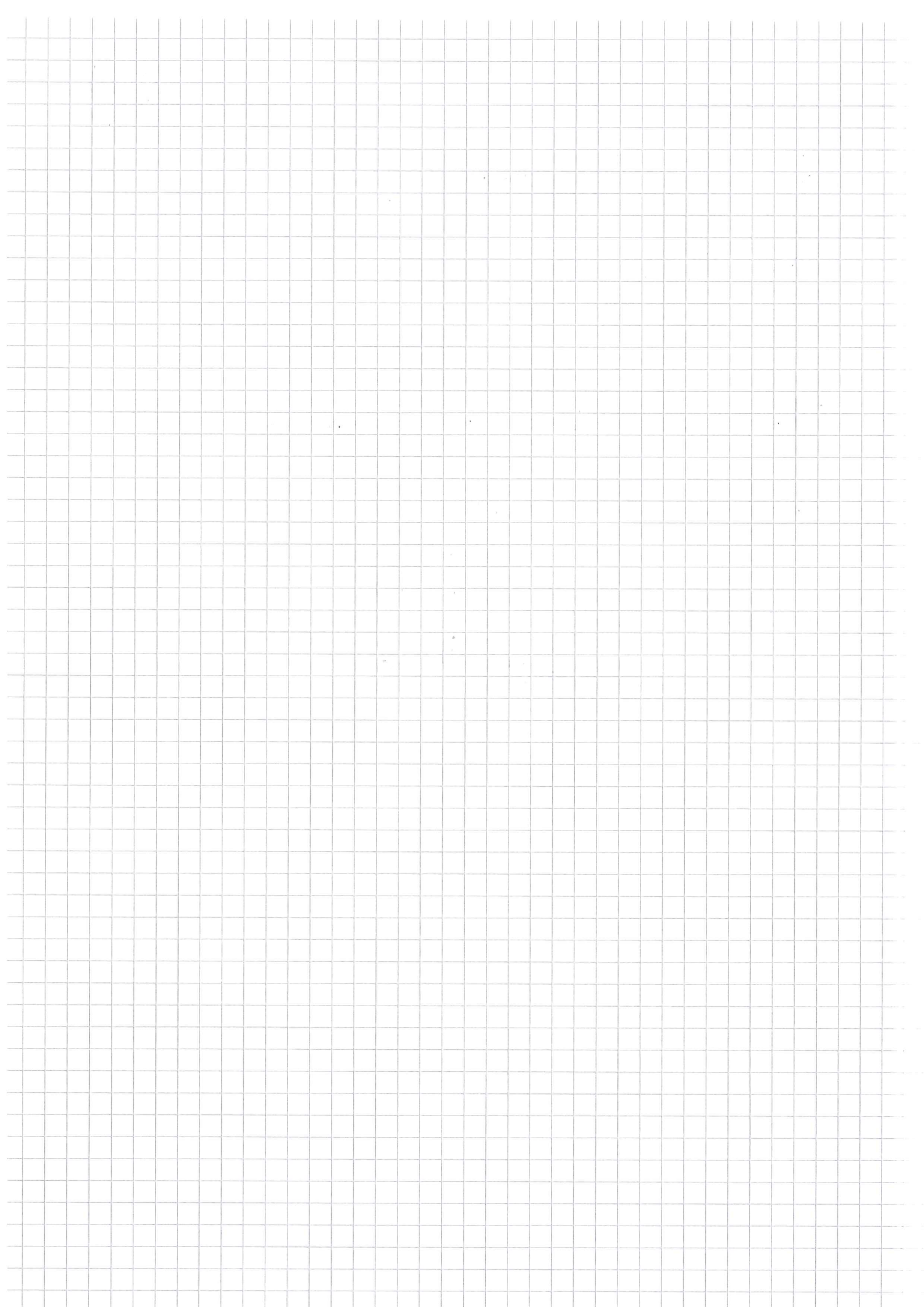


12)

H^{-1} is the dual space
of H_0^1 .

The formal, but less instructive
definition is that

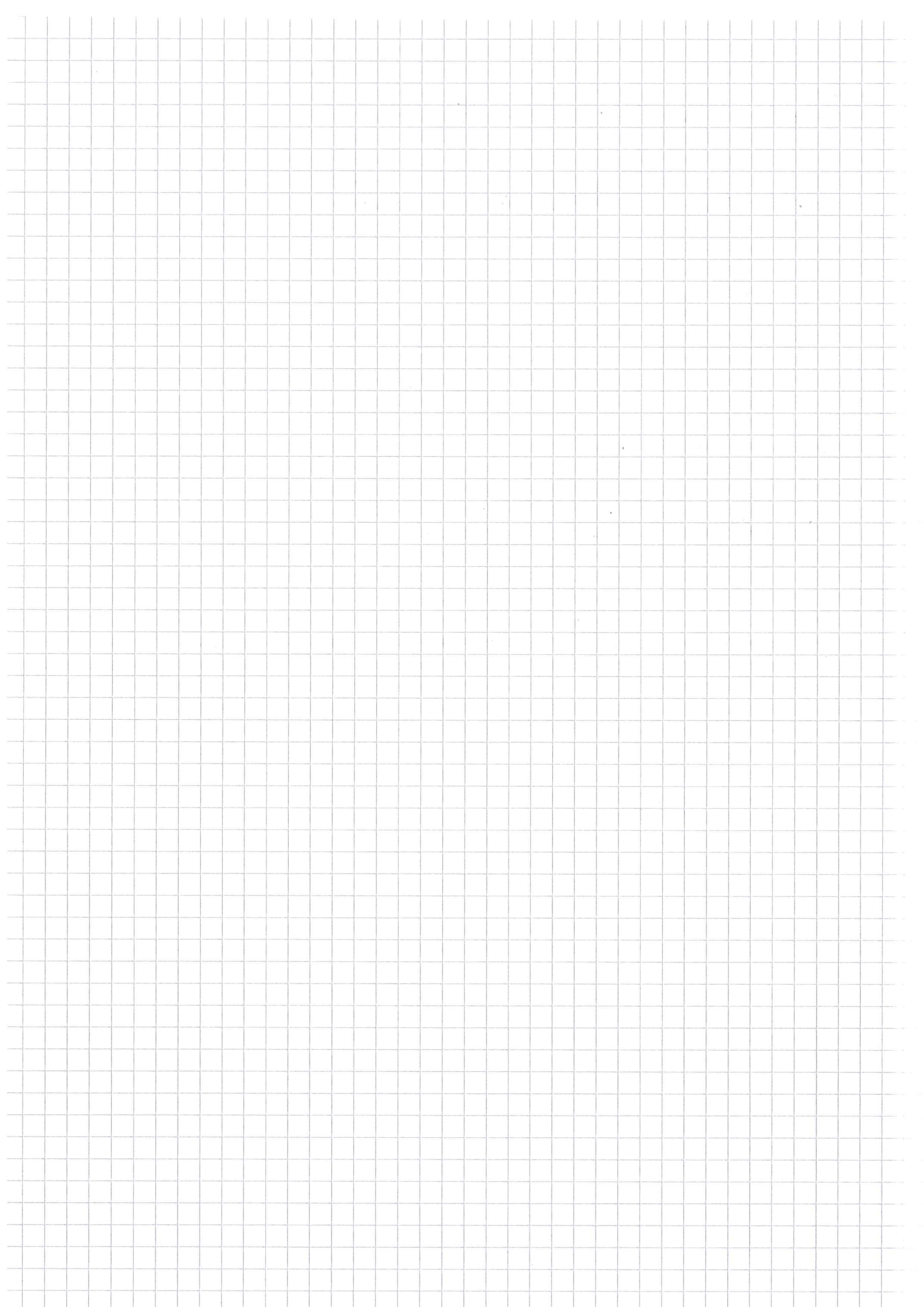
$$\|f\|_{-1} = \sup_{v \in H_0^1} \frac{(f, v)}{\|v\|_1}$$



There is a distinction
between functions and
functionals.

A functional is
something that takes a
function and produce a number,

~~A.~~



Any function can together
with the L^2 inner product

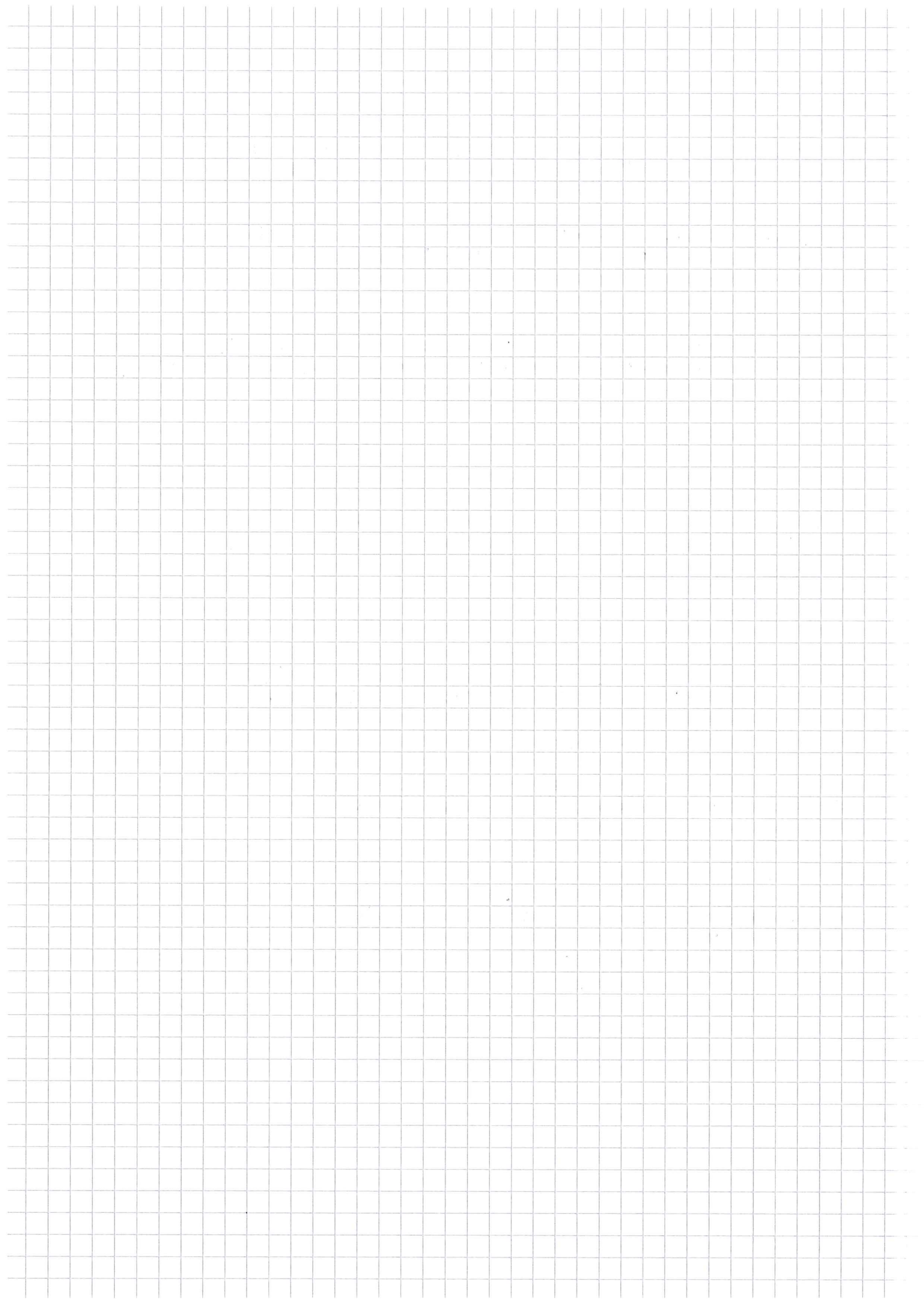
be turned into a functional! \square

Let f be some function.

Then the corresponding functional

is when applied to v

$$f(v) = \int_{\Omega} f v \, dx.$$



Hence, in some sense

the ~~distin~~ distinction

between functions and

functionals are not that

important in practice.

But functionals may be

defined for "generalized functions"

such as Dirac's delta.

