Lecture 10

Space before time in the discretization of Navier - Stokes.

Last time we arrived at the following system

1) $M\dot{u}_n + K(u_n)u_n = -Qp_n + Au_n + b$

2) $Q^T u = 0$

This is a DAE (differential - algebraic equation)

M is the mass matrix

$$M_{ij} = \int_\Omega N_i N_j$$

A stiffness matrix

$$A_{ij} = \int_\Omega \nabla N_i : \nabla N_j$$

$$Q_{ij} = \int \nabla N_i \cdot L_j$$

is discrete divergence

etc.

When we started with time first, we solved for the tentative velocity first, the a pressure update.

Let us do the same now

1. $$Mu^* = Mu^l + \Delta t\left(-K(u^l)u^l - Qp^l + Au^l + f^l\right)$$

$$u^{l+1} = u^* + u^c$$

u^{l+1} should satisfy

2) $$M u^{l+1} = M u^l + \Delta t \left(K(u^l)\, u^l - Q p^{l+1} + A u^l + f^l \right)$$

3) $$Q^T u^{l+1} = 0$$

b)

As before we subtract

equation 4) -2) from 3) -1)

change order

$\Rightarrow$

$$M u^{l+1} - M u^{*} = -\Delta t\left(Q p^{l+1} - Q p^{l}\right)$$

Using the fact that

$$Q^{T} u^{l+1} = 0$$

From

$$Mu^{l+1} - Mu^{*} = -\Delta t\left(Qp^{l+1} - Qp^{l}\right)$$

$$u^{l+1} = u^{*} - \Delta t M^{-1} Q \underbrace{\left(p^{l+1} - p^{l}\right)}_{\phi}$$

$$u^{l+1} = u^{*} - \Delta t M^{-1} Q \phi$$

$\Longrightarrow$

$$0 = Q^{T} u^{l+1}$$

$$= Q^{T} u^{*} - \Delta t Q^{T} M^{-1} Q \phi .$$

In other words

$$\Delta t \, Q^T M^{-1} Q \, \varphi = Q^T u^*$$

We remember that when we did it earlier, we arrived at

$$\Delta t \, \Delta \varphi = \nabla \cdot u^*$$

What is the difference ?

- Q^T is a discrete divergence
- Q is a discrete gradient
- M is a discrete (scaled) identity.

[Note M does not scale as 1, but rather h^d - d is dimension.]

Hence

$Q^T M^{-1} Q$ looks like

a discrete divergence *

multiplied by an inverse

(scaled) identity operator multiplied

with a discrete gradient

$$\nabla \cdot I^{-1} \nabla \sim \Delta \text{ or } \nabla^2$$

A difference is that these operators/matrices already have boundary conditions.

[No extra boundary conditions required !]

More generally

Let us consider a system

1) $N u^{l+1} + \Delta t\, Q p^{l+1} = q$

2) $Q^T u^{l+1} = 0$

Here N is

$M + \Delta t \left(-K(u^l) + A\right)$

[Hence, a semi-implicit discretization in time, after the space discretization]

Then, by the same argument as before a tentative guess

1) $N u^* + \delta t Q p^l = q$

$\Rightarrow$ Subtracting 11) - 1) from 1)

switch order

$\Rightarrow N u^{l+1} - N u^* + \cancel{\delta t Q p^{l+}}$

$+ \delta t Q (p^{l+1} - p^l) = 0$

i.e

$$u^{l+1} = u^* - \Delta t N^{-1} Q (\phi)$$

$$Q^T u^{l+1} = 0$$

$$\Rightarrow$$

$$Q^T N^{-1} Q \phi = -\frac{1}{\Delta t} Q^T u^*$$

Conclusion :

Also when we discretize in space before time, we can do projection steps.

A difference is that boundary conditions are already in the matrices.

~~As~~ ~~Such~~

Are non-physical oscillations bad?

Consider the problem

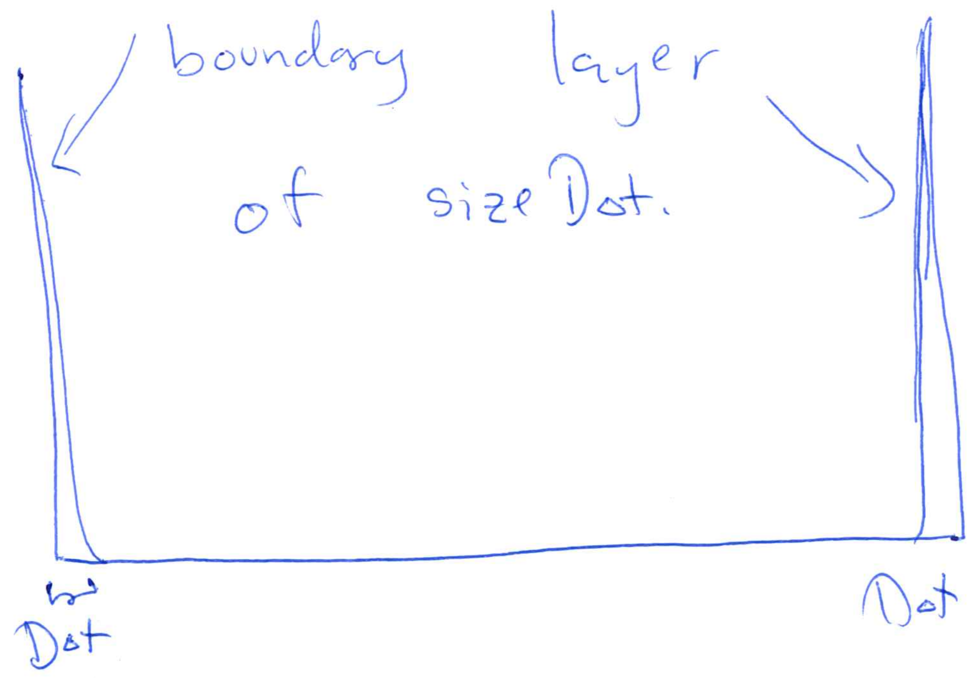
$$u_t = D \Delta u$$

$$u(x,0) = 0$$

$$u(0,t) = u(1,t) = 1$$

Solution at first timestep

with some small Δt



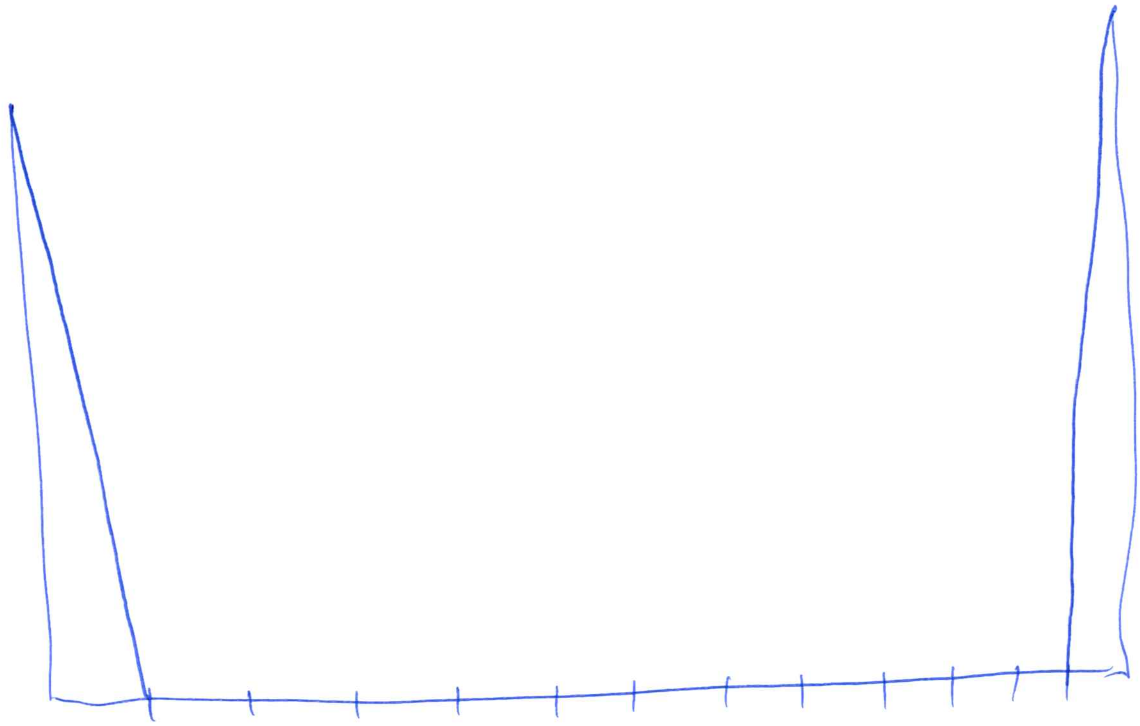
I can form a standard Galerkin FEM method, or a version with a lumped mass matrix. (similar to SUPG in many ways)

lumped mass matrix

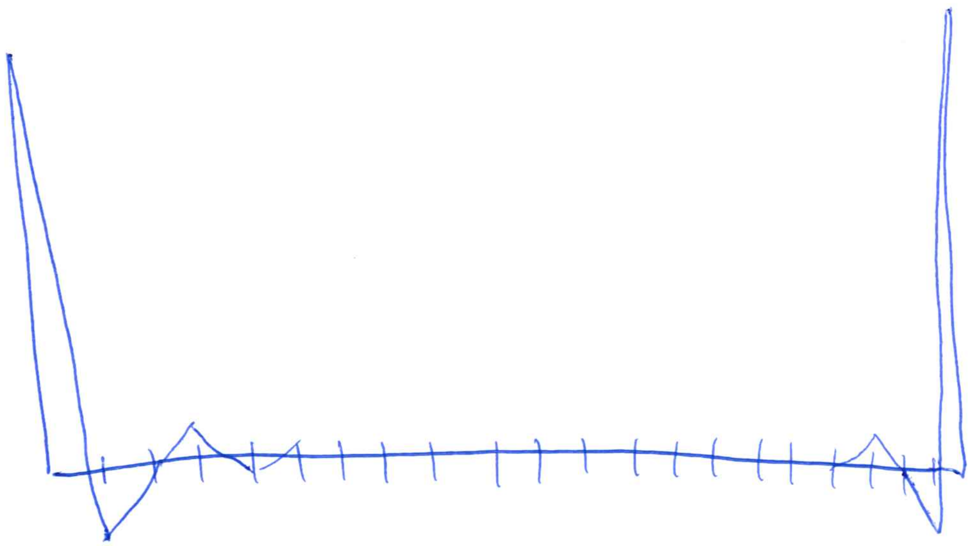
$$M_{ii} = \cancel{\text{sum}} \sum_j M_{ij} \quad \left(\text{or } \sum_j |M_{ij}|\right)$$

$$M_{ij} = 0.$$

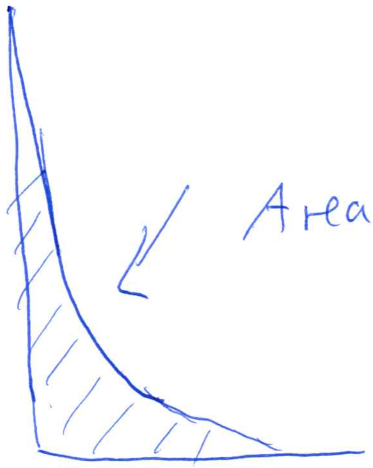
Coarse mesh solution
with lumped



Coarse mesh solution
with std Galerkin

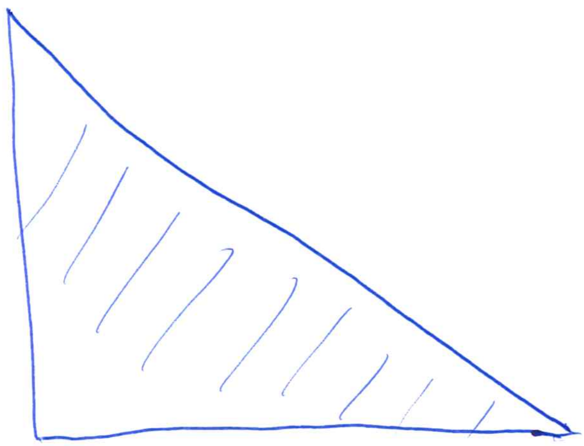


(zooming in)

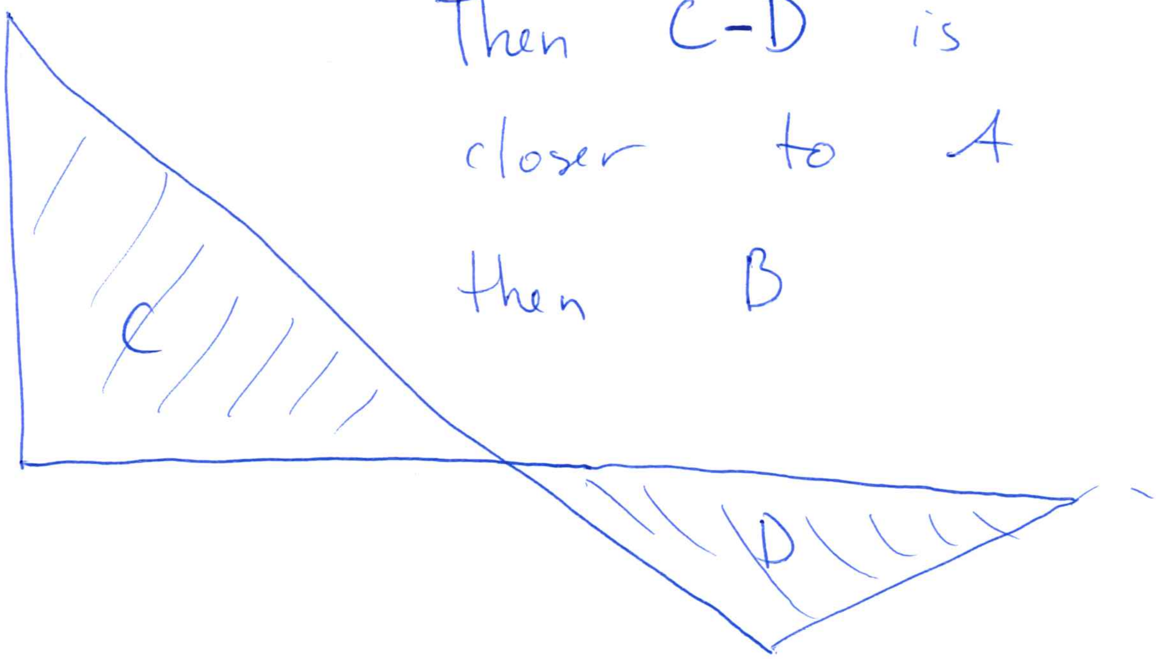


Area under curve : A

lumped



Area under curve : B



Then C-D is closer to A then B

In the book

chapter we see

much better "long time"

approximation with std.

Galerkin