Lecture 4

Last time we arrived at
a finite element problem:

Find $u_h \in V_{h,g}$ such that

$$a(u_h, v_h) = L(v_h) \quad \forall v_h \in V_{h,0}$$

which is an approximation of
the continous problem

Find $u \in V_g$ such that

$$a(u, v) = L(v) \quad \forall v \in V_0$$

I claimed that

$$\| u - u_h \|_1 \leq c h^k \| u \|_{k+1}$$

without being very spesific

First of all

$$\|u\|_{k+1} = \|u\|_{H^{k+1}(\Omega)}$$

$$= \left(\sum_{j \leq k+1} \left\| \left(\frac{\partial}{\partial x}\right)^j u \right\|^2_{L^2(\Omega)} \right)^{1/2}$$

Hence,

$\|u\|_0$ involves no derivatives and equals L^2

$\|u\|_1$ involves first order derivatives

$\|u\|_2$ involves second order derivatives

$\|u\|_3$ involves third order derivatives.

In 2D:

$$\frac{\partial^3 u}{\partial x^3}, \frac{\partial^3 u}{\partial x \partial y^2}, \frac{\partial^3 u}{\partial x^2 \partial y}$$

etc.

<u>Taylor series</u> :

$$T_n(x) = f(a) + f'(a)(x-a) + \dots + \frac{f^n(a)}{n!}(x-a)^n$$

Then for $y \in (x,a)$

$$|f(y) - T_n(y)| \leq \max_{z \in (x,a)} \frac{f^{n+1}(z)}{(n+1)!}(z-a)^{n+1}$$

Hence, a Taylor series using a polynomial order n is an $n+1$ ~~appr~~ order approximation, given that $z-a < 1$.

Here, we considered max-norm or L^∞.

Bramble – Hilbert lemma.
(Clement, Scott - Zang)

Let Ω be a bounded domain with diameter d.
(Think of Ω as an element)

Then if P is a space of polynomials of order k there exists a v in P such that

$$(*) \quad \| u - v \|_{H^k(\Omega)} \leq C d^{m-k} \| u \|_{H^m(\Omega)}$$

[there are conditions on m and k]

For $k=0$ we get something similar to Taylor's result.
In fact, the above is due to Taylor.

Results like 4)

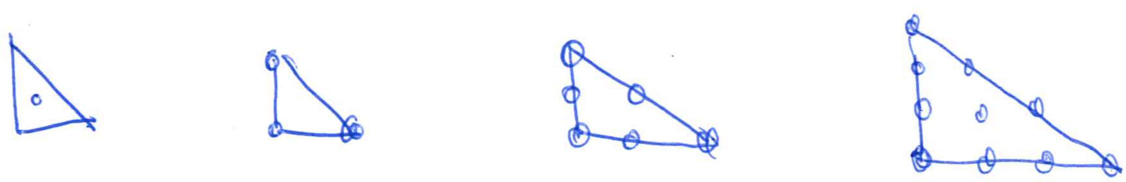
is not unique to finite elements.

Similar results can be found in splines, Fourier analysis neural networks.

A ~~unique~~ convenient ~~feature~~ feature of polynomial approximation is that I can explicitly construct v. by nodal interpolation.

In 2D

$k=0$ $\quad$ $k=1$ $\quad$ $k=2$ $\quad$ $k=3$



Lagrange 1 $\quad$ Lagrange 2 $\quad$ Lagrange 3

So, $v = I_h u$ where I_h is the nodal interpolation at the points shown on the last slide.

There is one big problem here!!

We do not know u!
We only know that
it is a solution of
a PDE problem.

Some properties of our PDE problem

Abstract version	LA version
$a(u,u) \geq \alpha \|u\|_1^2$	
$a(u,v) \leq C \|u\|_1 \|v\|_1$	
$L(u) \leq D \|v\|_1$	

Abstract version

$a(u,u) \geq \alpha \|u\|_1^2 \quad \forall u$ [illegible] 1)

$a(u,v) \leq C \|u\|_1 \|v\|_1 \quad \forall u, v$ 2)

$L(v) \leq D \|v\|_1$ 3)

Linear algebra problem : $Ax = b$. (**)

Does ~~(**)~~ have a unique, "well-posed" solution.

Well-posed means here <u>not ∞</u>

$x^T A x > 0$

$x^T A y < C \|x\| \|y\|$
(no entry is ∞)

$b < \infty$

Alt

$x^T A x \geq \alpha \|x\|^2$

$x^T A y \leq C \|x\| \|y\|$

$b \leq D$

for α, C, D positive numbers.

Some comments concerning the linear algebra thinking.

$Ax=b$ is not one problem with one matrix.

We have familes of matrices and recipes to make them on given meshes.

We commonly write

$$A_h u_h = b_h$$

and consider that $h \to 0$.

Galerkin orthogonality

We have u, u_n such that

$$a(u,v) = L(v) \quad \forall\ v \in V$$
$$a(u_n, v_n) = L(v_n) \quad \forall\ v_n \in V_n$$

Furthermore $V_n \subset V$

$\Rightarrow$ Hence $a(u, v_n) = L(v_n)\ \forall\ v_n \in V_n$

and by direct consequence.

$$a(u - u_n, v_n) = L(v_n) - L(v_n) = 0$$

Cool ! The error is orthogonal (in some sense) to any v_n in V_n.

Wow !

Lets see what we can do.

From 8) - 1) we have

Galerkin orthogonality

$$\alpha \|u-u_n\|_1^2 \leq a(u-u_n, u-u_n)$$

$$\leq a(u-u_n, u-v+v-u_n)$$

$$= a(u-u_n, u-v) + \cancel{a(u-u_n, \underbrace{v-u_n}_{\in V})}^{=0}$$

$$\leq C\|u-u_n\|_1\|u-v\|_1$$

8) - 2)

$$\Longrightarrow \alpha\|u-u_n\|_1^2 \leq C\|u-u_n\|_1\|u-v\|_1$$

$$\Longrightarrow \|u-u_n\|_1 \leq \frac{C}{\alpha}\|u-v\|_1$$

In conclusion then

For solving the problem, e.g

$$-\Delta u = f \qquad \text{with} \quad a(u,v) = \int_\Omega \nabla u \cdot \nabla v$$

$$u = 0 \qquad\qquad L(v) = \int_\Omega f v$$

We have, for the FEM solution u_h

$$\|u - u_h\|_1 \leq \frac{C}{\alpha} \|u - v\|_1 \quad \text{for any } v \text{ in } V_h$$

Thus, I can choose $v = I_h u$

[even though I don't know u, but I use it as a theoretical tool]

$$\Rightarrow \|u - u_h\|_1 \leq \frac{C}{\alpha} h^{m-k} \|u\|_{H^m}$$