Why function spaces ? 1)

Properties :

- if u, v are functions in V then $w = u + v$ is also in V

- if $u \in V$ and $c \in \mathbb{R}$ then $w = cu \in V$.

- if V is spanned by $\{N_i\}_{i=0}^{N}$ then $u = \sum_{i=0}^{N} u_i N_i$ is a typical function in V.

Function spaces have nice properties

- Cauchy-Schwarz inequality

If V is a Hilbert space it involves an inner product $(\cdot, \cdot)$

Then

$$(u, v) \leq \|u\|_V \|v\|$$

- We use function spaces that are complete (Hilbert, Banach)

Hence, if $v_n \xrightarrow[n\to\infty]{} v$

and $u_n \in V$ then

$u \in V$.

Our function spaces typically have approximation properties.

Given a family of meshes Ω_h that approximate Ω.

Let V be the continuous function space while $\{V_h\}$ is the discrete spaces.

Typically we can then say that for any $u \in V$ we have

$$\|u - u_h\|_V \leq c h^{\alpha} \|u\|_V$$

Fundamental lemma (Gass-Green)

(integration by parts)

$$\int_{\Omega} -\nabla \cdot (K \nabla u) v \, dx = \int_{\partial\Omega} -K \frac{\partial u}{\partial n} v \, ds$$

$$+ \int_{\Omega} K \nabla u \cdot \nabla v \, dx$$

As already mentioned, this is incredibly useful and the basis for a lot of results.

Deriving a variational formulation

3 Step

1. multiply by a test function

2. integrate over the domain

3. integrate by parts.

Poisson problem:

$$-\nabla\cdot(K\nabla u) = f \quad \text{in } \Omega \qquad (1)$$

$$u = u_0 \quad \text{on } \partial\Omega_D \qquad (2)$$

$$-K\nabla u\cdot n = g \quad \text{on } \partial\Omega_N \qquad (3)$$

Step 1.

$$-\nabla\cdot(K\nabla u)\cdot v = f\cdot v$$

[You may ask whether you should hit (2) and (3) with a test function as well. Sometimes this is useful.]

Step 2.

$$\int_\Omega -\nabla\cdot(K\nabla u)\, v \, dx = \int_\Omega f v \, dx$$

$$\parallel$$

Step 3.

$$\int_\Omega K\nabla u\cdot\nabla v - \int_{\partial\Omega} K\frac{\partial u}{\partial n} v \, ds$$

Here, we know

$K\frac{\partial u}{\partial n}$ on $\partial\Omega_N$ by (3)

What about $\partial\Omega_D$ and $u = u_0$?

On $\partial\Omega_D$ we set $u = u_0$ and let $v = 0$. This happens in bc.apply( ) in FEniCS in the sense that $u = u_0$ is put in pointwise at every point on the boundary.

Summarizing we have that

$$\int_{\partial\Omega} k \frac{\partial u}{\partial n} v \, ds = \int_{\partial\Omega_D} k \frac{\partial u}{\partial n} \underbrace{v}_{=0} \, ds + \int_{\partial\Omega_N} \underbrace{k \frac{\partial u}{\partial n}}_{g} v \, ds$$

$$= \int_{\partial\Omega_N} g v \, ds$$

The variational formulation

Find $u \in V_g$ such that

$$a(u,v) = L(v) \quad \forall \; V_0$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$$

$$L(v) = \int_\Omega f v \, dx + \int_{\partial\Omega_N} g v \, ds$$

Comments

$u \in V_g$ determines the number of degrees of freedom / unknows

$v \in V_0$ determine the number of equations.

If $V_g = V_0$ except on the Dirichlet boundary we call it the Galerkin method. We then have that

$\dim(V_g) = \dim(V_0)$

→ "well defined" problem in terms of a rectangular matrix

FEM version

Find $u_h \in V_{g,h}$ such that

(*) $$a(u_h, v_h) = L(v_h) \quad \forall \; v \in V_{0,h}$$

The spesific form of (*) that wi use is

$$u_h = \sum u_i N_i$$

FEniCS Function

FEniCS Trial Function

$$v_h = N_j$$

This is then written as a linear system.

$$Ax = b$$

$$A_{ij} = a(N_i, N_j)$$

$$b_j = L(N_j)$$

$$x_i = u_i$$

Notice that

$$Ax = \sum a(N_i, N_j) u_i$$

$$= a(\sum u_i N_i, N_j)$$

$$= a(u_h, N_j) = L(N_j)$$

Useful function spaces

L^2 : consist of functions such that

$$\int_\Omega u^2 \, dx < \infty$$

: Inner product between u, v

$$(u, v)_{L^2} = \int_\Omega u v \, dx$$

H^1 : consists of functions such that

$$\int_\Omega (u^2 + (\nabla u)^2) \, dx < \infty$$

Inner product ~~between~~ between u, v

$$(u, v)_{H^1} = \int_\Omega (uv + \nabla u \cdot \nabla v) \, dx$$

Result that we want to prove.

u is the continuous solution.

u_h is the family of solutions parameterized by h.

Then

$$\| u - u_h \|_1 \leq c h^\alpha \| u \|_\alpha$$

Question:

What is the cool thing with an inner product?