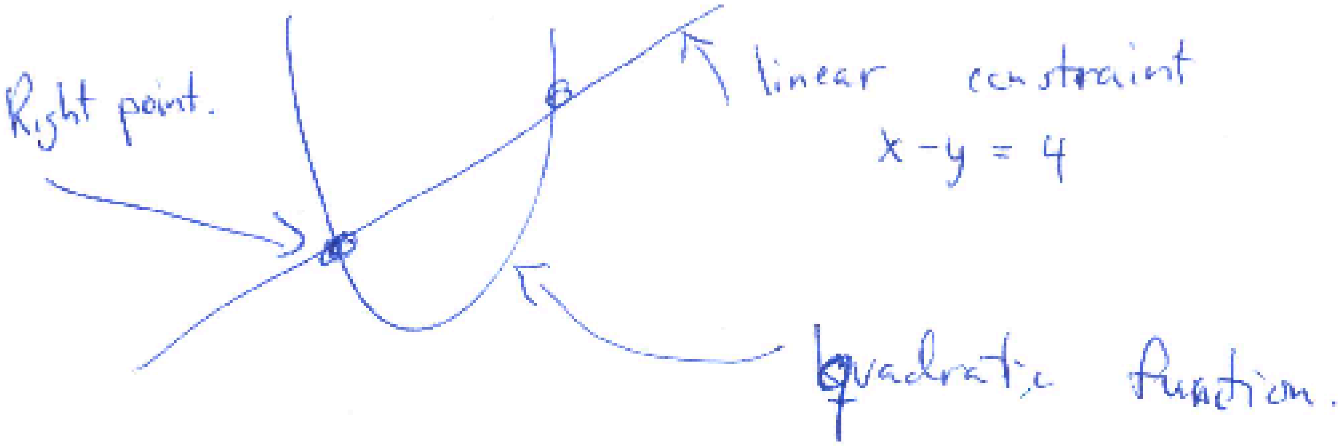
The method of Lagrange multipliers.

Consider the problem

$$\text{minimize } (x^2+y^2-7) \qquad 1)$$
$$\text{subject to } x-y=4.$$

Intuitively, if we were in 1D, the situation is



The Poisson problem
with homogenous Neumann boundary
conditions is

$$-\Delta u = f \quad \text{in } \Omega$$

$$\frac{\partial u}{\partial n} = g$$

It can be stated as a
minimization problem

$$L(u) = \frac{1}{2}\int_{\Omega} \nabla u^2 - \int_{\Omega} f u - \int_{\partial\Omega} g u \longrightarrow \min$$

Here L is a quadratic
functional in u.

3)

We have that

$L(v)$ ~~...~~ attains its minimum

for u when

$$\frac{\partial L}{\partial u} = 0$$

$$0 = \frac{\partial L}{\partial u} = \int_{\Omega} \nabla u \cdot \nabla v - \int_{\Omega} f v - \int_{\partial\Omega} g v$$

The problem with the homogenous Neumann problem is that it is singular, ~~what~~ with constant functions in the kernel.

We would therefore like to add a <u>linear</u> constraint to the <u>quadratic</u> functional.

The linear constraint "should" be

$$\int_\Omega u \, dx = 0$$

For problem 1, ie

$$x^2 + y^2 - 7 \to \min$$

subject to $x - y = 4$

The Lagrange multiplier method produce the Lagrangian functional

$$L(x, y, \lambda) = x^2 + y^2 - 7 + \lambda(x - y - 4).$$

Further the solution is attained for the equation system

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial z} = 0.$$

Similarly the Lagrangian functional for the Neumann problem becomes

$$L(u,\lambda) = \frac{1}{2}\int_\Omega (\nabla u)^2 - \int_\Omega f u - \int_{\partial\Omega} g u$$

$$+ \int_\Omega \lambda u \quad \leftarrow \quad \boxed{\int \lambda u = \lambda \int u \text{ since } \lambda \in \mathbb{R}}$$

and the solution of the problem can be derived as

$$\frac{\partial L}{\partial u} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \lambda} = 0$$