# Elliptic equations and error estimation

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1. General - Elliptic equations

2. Maths - Lax-Milgram theorem

3. Numerics - Céa’s lemma

# 1. General - Elliptic equations

# Characterisation of linear, 2nd order PDEs

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + d(x,y)\frac{\partial u}{\partial x} + e(x,y)\frac{\partial u}{\partial y} = f(u,x,y)$$

It can be shown that for a linear PDE, the existence of characteristics depends on the sign of $\Delta \triangleq b^2 - 4ac$:

- $\Delta > 0$ : Characteristics: **hyperbolic equation** *(information travels at some finite speed)*
- $\Delta < 0$ : No characteristics: **elliptic equation** *(information travels infinitely fast)*
- $\Delta = 0$ : Degenerate case: **parabolic equation**

<u>Elliptic equations are nice:</u>

- The behaviour at one point influences the behaviour everywhere, the solution si expected to be **smooth**
- They are strictly boundary values problems (no initial conditions)

# Generalization: ellipticity condition in dimension $n$

- We consider a family of functions $(a_{ij})$ satisfying the ellipticity condition: $\displaystyle\sum_{i,j}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2$
  ($\alpha > 0$, $\xi \in \mathbb{R}^n$ )

- An elliptic equation is an equation for $u$ of the form: $\displaystyle a_0 u - \sum_{i,j}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) = f$

Note the link with the coercivity condition for the biliniear form $a$ in Lax-Milgram

## Example 1/3: Poisson's equation (steady-state diffusion)

$$\nabla^2 u = f$$

Canonical example of hyperbolic equation

## Example 2/3: Steady-state advection-diffusion equation with divergence-free velocity field

$$\nabla^2 u - (w \cdot \nabla)\, u = 0$$

# Example 3/3: Linear elasticity

- Small displacements, linearisation of the strain tensor: $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$

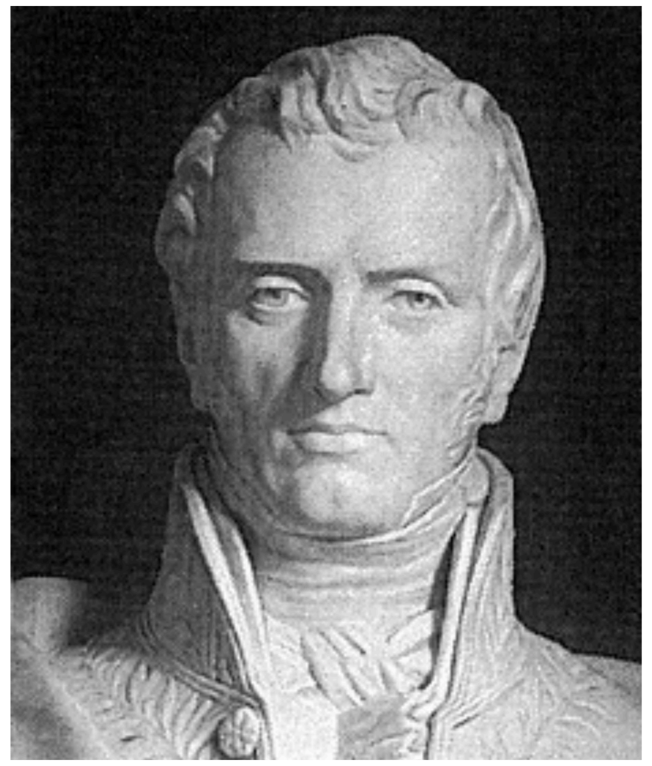
- Hooke's law
  (for homogeneous and isotropic materials) $\sigma_{ij} = C_{ijkl}e_{kl} = \lambda\delta_{ij}e_{kk} + 2Ge_{ij}$

- Mechanical equilibrium
  (no inertia) $\nabla \cdot \sigma + F = 0$

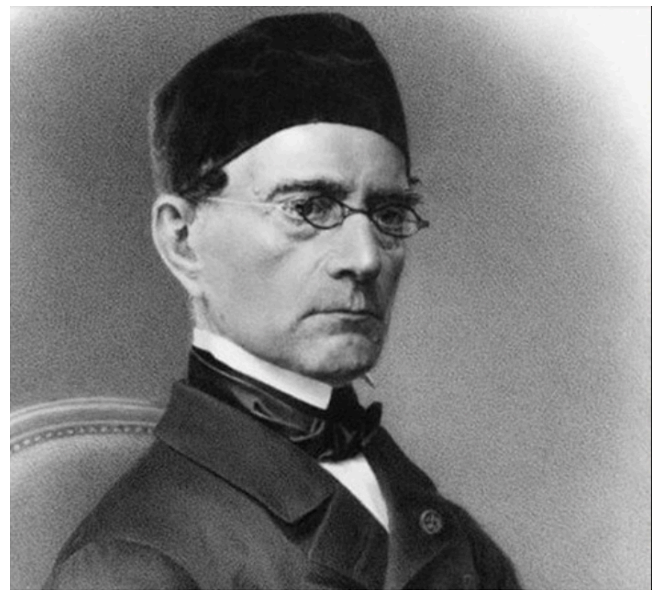
$$(\lambda + G)\,\nabla\left(\nabla \cdot u\right) + G\nabla^2 u + F = 0$$

We'll prove the elliptic condition later

Hooke
1635-1703



Navier
1785-1836



Lamé
1795-1870

# 2. Maths - Lax-Milgram theorem

# Lax-Milgram theorem

Let $V$ be a Hilbert space

Let $a : V \times V \rightarrow \mathbb{R}$ be bilinear and :

- continuous, i.e. there is a constant $M > 0$ such that, for any $u$ and $v$ in $V$ :

$$|a(u,v)| \leq M \|u\|_V \|v\|_V$$

- coercive (elliptic), i.e. there is a constant $\alpha > 0$ such that, for any $u$ in $V$ :

$$|a(u,u)| \geq \alpha \|u\|_V^2$$

Let $b : V \rightarrow \mathbb{R}$ be linear and :

- continuous, i.e. there is a constant $L > 0$ such that, for any $v$ in $V$ :

$$|b(v)| \leq L \|v\|_V$$

We take $V \subset H^1(\Omega)$
+ conditions at boundaries discussed next

$$H^1(\Omega) = \{u,\ \partial_i u \in L^2(\Omega)\}$$

$$< u, v >= \int_\Omega uv + \int_\Omega \nabla u \cdot \nabla v$$

$$\|u\|_{H^1} = \left(\int_\Omega u^2 + \int_\Omega \nabla u \cdot \nabla u\right)^{1/2}$$

Lax-Milgram theorem states that :

There is a unique solution $u$ to the following problem: $a(u,v) = b(v) \quad \forall v \in V$

# Recipe to prove that Lax-Milgram theorem applies:

$$PDE(u)$$

1. Multiplying by a test function $v$ and integrating over the domain

2. Integrating by part

3. Applying boundary conditions

$$a(u, v) = b(v) \quad \forall v$$

Using some maths (Cauchy-Schwarz, Poincaré, Korn) we need to check that:

- $a$ is continuous and coercive
- $b$ is continuous

# Generalized Poincaré's inequality

If $\Omega$ is a connected subset of $\mathbb{R}^n$, then there is a positive constant $C$ such that
for all $u \in \tilde{H}_0^1(\Omega) = \{u \in H^1, u = 0 \text{ on } \Gamma_1 \subset \Omega\}$ where the measure of $\Gamma_1$ is non-zero

$$\|u\|_{L^2} \leq C\|\nabla u\|_{L^2}$$

where the constant $C$ only depends on $p$ and $\Omega$.

**This shows that $\|(\,.\,)\|_{H^1}$, and $|(\,.\,)|_{H^1}$ are equivalent norms, and only if we have a Dirichlet condition on $u$**

# Diffusion equation

$$\nabla^2 u = f \qquad \begin{aligned} u &= 0 \text{ on } \Gamma_1 \\ \nabla u \cdot n &= g \text{ on } \Gamma_2 \end{aligned}$$

Can we write it as a problem of the following form: $a(u,v) = b(v) \quad \forall v \in V$ **?**

(1) $$\int_\Omega \nabla^2 u v = \int_\Omega f v$$

(2) $$\int_\Omega \nabla u \nabla v = \int_\Omega f v + \int_{\Gamma_1} (\nabla u) \cdot n v + \int_{\Gamma_2} (\nabla u) \cdot n v$$

We take $u, v \in \tilde{H}_0^1(\Omega)$ $$\tilde{H}_0^1(\Omega) = \{f \in H^1(\Omega), f = 0 \text{ on } \Gamma_1\}$$

(3) $$\boxed{\int_\Omega \nabla u \nabla v = \int_\Omega f v + \int_{\Gamma_2} g v}$$

$$a(u,v) = b(v)$$

# Coercivity condition:

**Conditions**

$$|a(u,v)| \leq M\|u\|_{H^1}\|v\|_{H^1}$$

$$|a(u,u)| \geq \alpha\|u\|^2_{H^1}$$

$$|b(v)| \leq L\|v\|_{H^1}$$

$$a(u,u) = \int_\Omega \nabla u \nabla u = \frac{1}{2}\int_\Omega \nabla u \nabla u + \frac{1}{2}\int_\Omega \nabla u \nabla u$$

$$\int_\Omega \nabla u \nabla u = \|\nabla u\|^2_{L^2} \geq \frac{1}{C^2}\|u\|^2_{L^2}$$

Using Poincaré's inequality

$$\implies a(u,u) \geq \frac{1}{2}\min\left(1, \frac{1}{C^2}\right)\int_\Omega (\nabla u^2 + u^2) = \frac{1}{2}\min\left(1, \frac{1}{C^2}\right)\|u\|^2_{H^1}$$

# Advection-diffusion equation

$$\nabla^2 u - w \cdot \nabla u = 0 \qquad u = 0 \text{ on } \Gamma$$

Can we write it as a problem of the following form: $a(u, v) = L(v) \ \forall v \in V$

$$(1) \qquad \int_\Omega \nabla^2 u v + \int_\Omega w \cdot (\nabla u) v = \int_\Omega f v$$

We take $u, v \in H_0^1(\Omega)$ $\qquad H_0^1(\Omega) = \{f \in H^1(\Omega), f = 0 \text{ on } \Gamma\}$

$$(2) \qquad \int_\Omega \nabla u \nabla v + \int_\Omega w \cdot \nabla(u) v = \int_\Omega f v$$

$$a(u, v) + c(u, v) = b(v)$$

# Coercivity condition:

We have already shown that the coercivity condition applies in case of $a(u, v) = b(v)$ .

We need to make sure that the same is the case for $a(u, v) + c(u, v) = b(v)$

= 0 because of incompressible flow

= 0 in case of only Dirichlet BC

$$c(u,v) = \int_\Omega w(\nabla u)v = -\int_\Omega (\nabla \cdot w)uv - \int_\Omega wu\nabla v + \int_\Gamma uvw \cdot n = -c(v,u)$$

Here we have used:

$$\int_\Omega (\nabla \cdot w)uv = -\int_\Omega w\nabla(uv) + \int_\Gamma uvw \cdot n = -\int_\Omega w\nabla(u)v - \int_\Omega w\nabla(v)u + \int_\Gamma uvw \cdot n$$

$$\implies c(u,u) = -c(u,u) = 0$$

# Linear elasticity (1/4)
# Weak problem

$$\sigma_{ij} = C_{ijkl} e_{kl}$$
$$e(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$$
$$\nabla \cdot \sigma + F = 0$$
$$u = 0 \text{ on } \Gamma_1$$
$$\tau \cdot n = g \text{ on } \Gamma_2$$

- Multiplying by a test function, integrating over the domain, integration by part: $\int_\Omega \sigma : \nabla v = \int_\Omega F \cdot v + \int_{\partial\Omega} (\sigma \cdot n) \cdot v$

- Using the symmetry of the stress tensor:

$$\begin{aligned} \sigma_{ij}\frac{\partial v_i}{\partial x_j} &= \frac{1}{2}\sigma_{ij}\frac{\partial v_i}{\partial x_j} + \frac{1}{2}\sigma_{ji}\frac{\partial v_i}{\partial x_j} \\ &= \sigma_{ij}\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \\ &= \sigma_{ij} e_{ij}(v) \end{aligned}$$

- Finally, after applying the boundary conditions:

$$\int_\Omega \sigma_{ij}(u) e_{ij}(v) = \int_\Omega F \cdot v + \int_{\Gamma_2} g \cdot v$$
$$a(u, v) = b(v)$$

# Linear elasticity (2/4)
# Lax-Milgram and coercivity

$$\sigma_{ij} = C_{ijkl} e_{kl}$$

$$e(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$$

$$\nabla \cdot \tau + F = 0$$

$$u = 0 \text{ on } \Gamma_1$$

$$\tau \cdot n = g \text{ on } \Gamma_2$$

$$\int_\Omega \sigma_{ij}(u) e_{ij}(v) = \int_\Omega F \cdot v + \int_{\Gamma_2} g \cdot v$$

$$a(u,v) = b(v)$$

Showing that $a$ and $b$ are continuous is not difficult, the tricky point is the coercivity of the bilinear form: $|a(u,u)| \geq \alpha \|u\|^2_{H^1}$ **?**

$$a(u,u) = \int_\Omega C_{ijkl} e_{ij} e_{kl}$$

$$\geq \alpha \int_\Omega e_{ij} e_{ij} = \sum_{i,j} \|e_{ij}\|_{L^2(\Omega)}$$

**If we assume that $C$ is such that $C_{ijkl} e_{ij} e_{kl} \geq \alpha e_{ij} e_{ij}$ for some positive constant $\alpha$**

# Linear elasticity (3/4)
# Korn's lemma

$$\sigma_{ij} = C_{ijkl} e_{kl}$$

$$e(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$$

$$\nabla \cdot \tau + F = 0$$

$$u = 0 \text{ on } \Gamma_1$$

$$\tau \cdot n = g \text{ on } \Gamma_2$$

$$\int_\Omega \sigma_{ij} e_{ij}(v) = \int_\Omega F \cdot v + \int_{\Gamma_2} g \cdot v$$

$$a(u,v) = b(v)$$

?

We need to prove: $$a(u,u) \geq \sum_{i,j} \|e_{ij}(u)\|_{L^2(\Omega)} \geq \alpha \|u\|_{H^1}^2$$

$$e(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$$

**Korn's inequality:**

There is a positive constant $C$ such that, for all $u$ in $[H^1(\Omega)]^3$:

$$\|u\|_{H^1} \leq C \left\{ \|u\|_{L^2}^2 + \sum_{i,j} \|e_{ij}(u)\|_{L^2}^2 \right\}^{1/2}$$

**One consequence, analogous to Poincaré's inequality:**

if the measure of $\Gamma_1$ is non-zero, then $\sum_{i,j} \|e_{ij}(u)\|_{L^2}^2$

defines a norm, and this norm is equivalent to the $H^1$ norm

# Linear elasticity (4/4)
# Physical interpretation of the need for Dirichlet

$$\sum_{i,j} \|e_{ij}(u)\|^2_{L^2}$$

defines a norm when the set over which the displacement is null is *big enough*

$$e(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$$

If that is a norm, in particular it means that: $e(u) = 0 \Leftrightarrow u = 0$

**This is wrong in general:** uniform translations and solid body rotations do not generate strains

**Constraining the displacements on enough points means that we remove these degrees of freedom to isolate a unique solution**

# 3. Numerics - Céa’s lemma

- We have seen 3 examples of equations where an ellipticity condition + Dirichlet conditions yields existence and uniqueness
- It can also give estimates of the numerical error

# Galerkin's method

(1871-1945)

- Lax-Milgram theorem gives a solvability condition for the following equation in $V$:

$$a(u, v) = b(v) \quad \forall v \in V$$

- It also works if we work in a subset $V_h \subset V$ (our finite element problem):

$$a(u_h, v_h) = b(v_h) \quad \forall v_h \in V_h$$

- This gives an interesting property:

$$a(u - u_h, v_h) = a(u, v_h) - a(u_h, v_h) = b(v_h) - b(v_h) = 0$$

Galerkin orthogonality: The projection error is orthogonal to $V_h$, $\quad a(u - u_h, v_h) = 0 \ \ \forall v_h \in V_h$

**We take $V_h$ the set of continuous functions that are polynomial (order $n$) on each cell of mesh size $h$**

# Céa's lemma

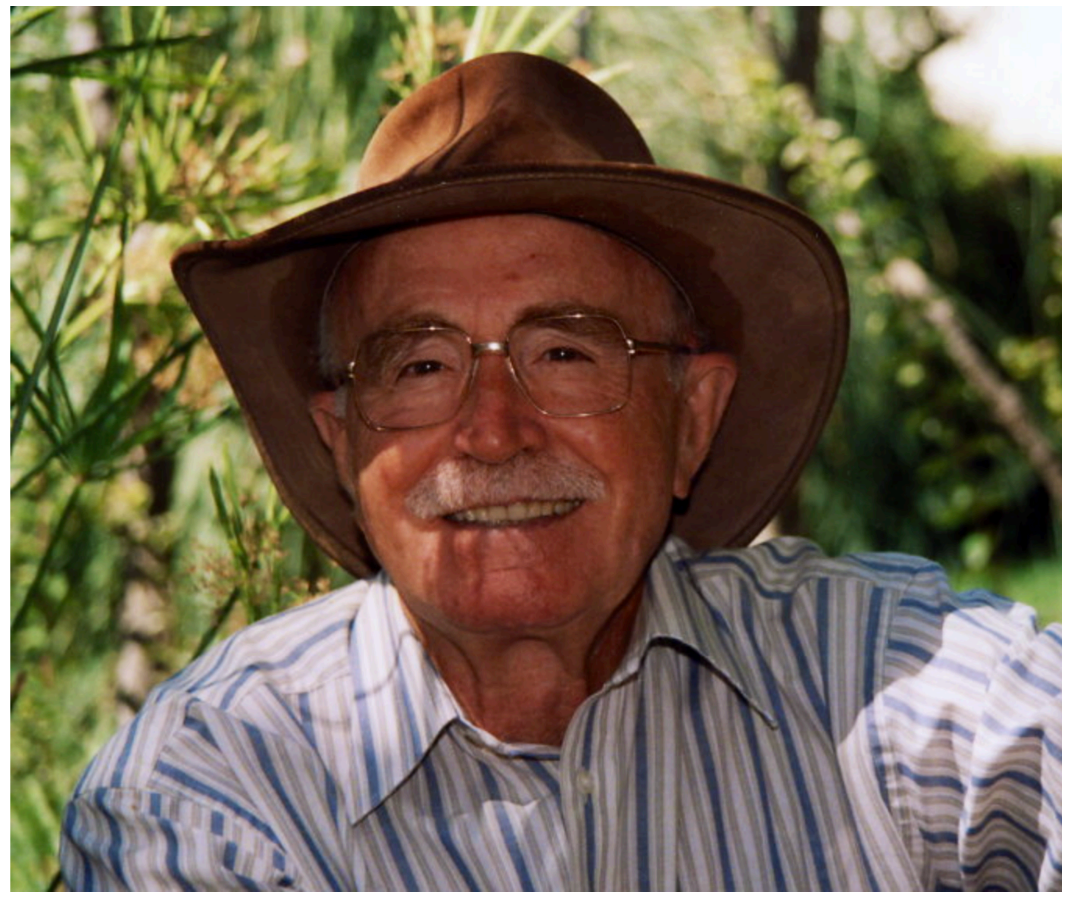
$$a(u,v) = b(v) \quad \forall v \in V$$

$$a(u_h,v_h) = b(v_h) \quad \forall v_h \in V_h$$

$$|a(u,u)| \geq \alpha \|u\|_V^2 \qquad (1)$$

$$|a(u,v)| \leq M \|u\|_V \|v\|_V \qquad (2)$$

$$a(u-u_h, v_h) = 0 \qquad (3)$$



(1964)

$$\alpha \|u-u_h\|_V^2 \leq a\left(u-u_h; u-u_h\right)$$ using coercivity (1)

$$\leq a\left(u-u_h; u\right) = a\left(u-u_h; u-v_h\right)$$ playing with Galerkin orthogonality (3)

$$\leq M \|u-u_h\|_V \|u-v_h\|_V$$ using continuity (2)

$$\|u-u_h\|_V \leq \frac{M}{\alpha} \|u-v_h\|_V \quad \forall v_h$$

# From Céa's lemma to practical estimations

solution of the continuous PDE

$$\|u - u_h\|_{H^1} \leq \frac{M}{\alpha}\|u - v_h\|_{H^1} \quad \forall v_h$$

solution of the finite element problem:

$$a(u_h, v_h) = b(v_h) \quad \forall v_h \in V_h$$

**We can take $v_h = \pi_h(u)$, the projection of $u$ on $V_h$**

**We take $V_h$ the set of continuous functions that are polynomial (order $n$) on each cell of mesh size $h$**

**Bramble-Hilbert lemma:** $\|u - \pi_h(u)\|_{H^n} \leq k h^{m-n}\|u\|_{H^m}$

$$\|u - u_h\|_{H^1} \leq C h^{m-1}\|u\|_{H^m}$$