7
Cartesian Tensors

7.1 Coordinate transformations

At the very beginning of this book vectors and scalars were defined as 'physical quantities'. But what does this mean mathematically? In this chapter a precise mathematical statement is developed, using the idea that the physical quantity exists independently of any coordinate system that may be used. This new mathematical definition of vectors and scalars is generalised to define a wider class of objects known as tensors. Throughout this chapter attention is restricted to Cartesian coordinate systems.

Consider a rotation of a two-dimensional Cartesian coordinate system x_1, x_2 through an angle θ (Figure 7.1) to give a new coordinate system x'_1, x'_2. Then by carrying out some simple geometrical constructions it can be seen that the coordinates of a point P in the x_1, x_2 system are related to those in the x'_1, x'_2 system by the equations

$$x'_1 = x_1 \cos\theta + x_2 \sin\theta, \tag{7.1}$$

$$x'_2 = x_2 \cos\theta - x_1 \sin\theta, \tag{7.2}$$

or in matrix form,

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

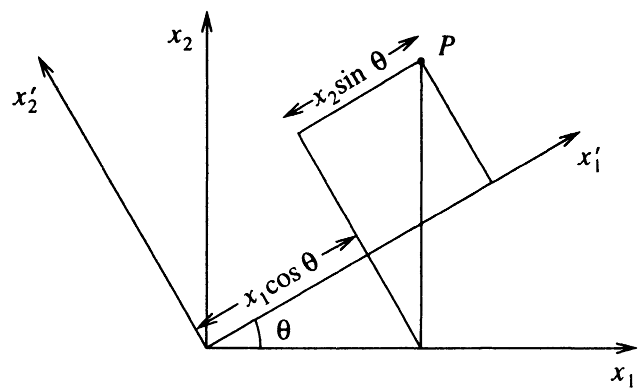

Fig. 7.1. Rotation of Cartesian coordinates through an angle θ.

The 2×2 matrix relating (x_1', x_2') to (x_1, x_2) will be referred to as L:

$$L = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{7.3}$$

The matrix multiplication can be written in suffix notation, since

$$\begin{aligned} x_1' &= L_{11}x_1 + L_{12}x_2 = L_{1j}x_j, \\ x_2' &= L_{21}x_1 + L_{22}x_2 = L_{2j}x_j, \end{aligned}$$

where the repeated suffix j implies summation, so

$$x_i' = L_{ij}x_j. \tag{7.4}$$

The rotation matrix L_{ij} has one particularly important property. The inverse of the matrix is a rotation through $-\theta$,

$$L^{-1} = \begin{pmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

which is the transpose of the matrix L. Thus $LL^T = I$, or in suffix notation, $L_{ij}L_{jk}^T = \delta_{ik}$. Since $L_{jk}^T = L_{kj}$, this can be written

$$L_{ij}L_{kj} = \delta_{ik}. \tag{7.5}$$

A matrix with this property, that its inverse is equal to its transpose, is said to be *orthogonal*. Using this property, the inverse of the transformation can be written down, simply by transposing the suffices:

$$x_i = L_{ji}x_j'. \tag{7.6}$$

Another important property of the matrix L is that its determinant is

$$|L| = \cos^2\theta + \sin^2\theta = 1.$$

So far we have only considered a two-dimensional rotation of coordinates. Consider now a general three-dimensional rotation. For a position vector $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3$, the i component in the dashed frame is defined by

$$x'_i = \boldsymbol{e}'_i \cdot \boldsymbol{x} = \boldsymbol{e}'_i \cdot \boldsymbol{e}_1 x_1 + \boldsymbol{e}'_i \cdot \boldsymbol{e}_2 x_2 + \boldsymbol{e}'_i \cdot \boldsymbol{e}_3 x_3 = \boldsymbol{e}'_i \cdot \boldsymbol{e}_j x_j.$$

This is of the form (7.4), where

$$L_{ij} = \boldsymbol{e}'_i \cdot \boldsymbol{e}_j, \tag{7.7}$$

so L_{ij} is the cosine of the angle between $\boldsymbol{e}'_i$ and $\boldsymbol{e}_j$. By the same argument, the matrix which transforms from the dashed frame to the undashed frame has i,j element $\boldsymbol{e}_i \cdot \boldsymbol{e}'_j = L_{ji}$, so again we see that the inverse of L is its transpose. Since $LL^T = I$, the determinant of L obeys $|L|^2 = 1$, so $|L| = \pm 1$. Orthogonal matrices with $|L| = 1$ represent rotations, while those with $|L| = -1$ are reflections.

From (7.4) and (7.6), two further important properties of L follow:

$$\frac{\partial x'_i}{\partial x_j} = L_{ij} \qquad \text{and} \qquad \frac{\partial x_i}{\partial x'_j} = L_{ji}. \tag{7.8}$$

7.2 Vectors and scalars

Now consider a vector $\boldsymbol{v}$. Its components transform from one coordinate system to another in the same way as the coordinates of a point, so

$$v'_i = L_{ij} v_j. \tag{7.9}$$

This equation gives the mathematical definition of a vector: $\boldsymbol{v}$ is a vector if its components transform according to the rule (7.9) under a rotation of the coordinate axes.

Similarly, a scalar s is defined by the property that its value is unchanged by a rotation of coordinates, so

$$s' = s. \tag{7.10}$$

Using these new definitions of scalars and vectors, in terms of their transformation properties under a rotation of coordinate axes, a number of rigorous results can be proved, as illustrated in the following examples. Suffix notation and the summation convention are used throughout.

Example 7.1

Suppose that $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors. Show that their dot product $\boldsymbol{a} \cdot \boldsymbol{b}$ is a scalar.

Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors, their components transform under rotation according to

$$a_i' = L_{ij}a_j, \qquad b_i' = L_{ij}b_j.$$

Now to show $\boldsymbol{a} \cdot \boldsymbol{b}$ is a scalar, we must show that its value in the dashed frame is the same as its value in the undashed frame.

$$(\boldsymbol{a} \cdot \boldsymbol{b})' = a_i'b_i' = L_{ij}a_jL_{ik}b_k = L_{ij}L_{ik}a_jb_k \tag{7.11}$$

$$= \delta_{jk}a_jb_k = a_kb_k = \boldsymbol{a} \cdot \boldsymbol{b}, \tag{7.12}$$

so $\boldsymbol{a} \cdot \boldsymbol{b}$ is a scalar.

Example 7.2

Suppose that f is a scalar field. Show that ∇f is a vector.

If f is a scalar then $f = f'$. To show that ∇f is a vector we need to determine how it transforms under a rotation of coordinates.

$$(\nabla f)_i' = \frac{\partial f'}{\partial x_i'} = \frac{\partial f}{\partial x_i'} = \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial x_i'}$$

using the chain rule. Now making use of (7.8),

$$\frac{\partial f}{\partial x_i'} = L_{ij}\frac{\partial f}{\partial x_j},$$

so ∇f obeys the transformation rule for a vector.

Example 7.3

A quantity is defined in a two-dimensional Cartesian coordinate system by $\boldsymbol{u} = (ax_2, bx_1)^T$. Show that this quantity can only be a vector if $a = -b$.

If $\boldsymbol{u}$ is a vector, it must transform according to the rule $u_i' = L_{ij}u_j$ where L_{ij} is the 2×2 rotation matrix (7.3). This gives

$$\boldsymbol{u}' = \begin{pmatrix} ax_2 \cos\theta + bx_1 \sin\theta \\ -ax_2 \sin\theta + bx_1 \cos\theta \end{pmatrix},$$

but from the definition of $\boldsymbol{u}$ we also have

$$\boldsymbol{u}' = \begin{pmatrix} ax_2' \\ bx_1' \end{pmatrix} = \begin{pmatrix} -ax_1 \sin\theta + ax_2 \cos\theta \\ bx_1 \cos\theta + bx_2 \sin\theta \end{pmatrix}.$$

By comparing these two expressions we can see that they only agree if $a = -b$, so this is the condition for $\boldsymbol{u}$ to be a vector.

7.3 Tensors

The definition of a vector as a quantity which transforms in a certain way under a rotation of coordinates can be extended to define a more general class of objects called tensors, which may have more than one free suffix. A quantity is a *tensor* if each of the free suffices transforms according to the rule (7.4). For example, consider a quantity T_{ij} that has two free suffices. This quantity is a tensor if its components in the dashed frame are related to those in the undashed frame by the equation

$$T'_{ij} = L_{ik}L_{jm}T_{km}. \tag{7.13}$$

The *rank* or *order* of the tensor is the number of free suffices, so the quantity T_{ij} obeying (7.13) is said to be a second-rank tensor. A tensor may have any number of free suffices. For example, a third-rank tensor P_{ijk} transforms according to the rule

$$P'_{ijk} = L_{ip}L_{jq}L_{kr}P_{pqr}. \tag{7.14}$$

The rule for a tensor of rank one is the same as the rule for a vector, so a vector can be regarded as tensor of rank one. Similarly, a scalar can be thought of as a tensor of rank zero.

We have already met one second-rank tensor, δ_{ij}, and a third-rank tensor, ϵ_{ijk}. Tensors can also be constructed from vectors, for example $\partial u_i/\partial x_j$ is a tensor. The demonstration that these quantities are indeed tensors is given in the following examples.

Example 7.4

Show that δ_{ij} is a tensor.

Consider the quantity $L_{ik}L_{jm}\delta_{km}$. From the substitution property of δ_{ij}, this is $L_{ik}L_{jk}$, which from the property (7.5) of L is δ_{ij}. Now $\delta'_{ij} = \delta_{ij}$, since δ_{ij} is defined the same way in any coordinate system. Thus δ_{ij} obeys the tensor transformation law, $\delta'_{ij} = L_{ik}L_{jm}\delta_{km}$.

Example 7.5

Show that ϵ_{ijk} is a tensor.

Since ϵ_{ijk} has three suffices, the appropriate transformation to consider is $L_{ip}L_{jq}L_{kr}\epsilon_{pqr}$. Using (4.10), this is $\epsilon_{ijk}|L| = \epsilon_{ijk}$, since $|L| = 1$ for a rotation. As for δ_{ij}, ϵ_{ijk} is defined in the same way in all coordinate systems so $\epsilon'_{ijk} = \epsilon_{ijk} = L_{ip}L_{jq}L_{kr}\epsilon_{pqr}$. Therefore ϵ_{ijk} is a third-rank tensor.

Example 7.6

If $\boldsymbol{u}$ is a vector, show that $\partial u_i/\partial x_j$ is a second-rank tensor.

Since $\boldsymbol{u}$ is a vector, $u'_i = L_{ik}u_k$.

$$\frac{\partial u'_i}{\partial x'_j} = L_{ik}\frac{\partial u_k}{\partial x'_j} = L_{ik}\frac{\partial u_k}{\partial x_l}\frac{\partial x_l}{\partial x'_j} = L_{ik}L_{jl}\frac{\partial u_k}{\partial x_l},$$

which is the transformation rule for a second-rank tensor.

7.3.1 The quotient rule

Tensors often appear as quantities relating two vectors, for example

$$a_i = T_{ij}b_j. \tag{7.15}$$

The quotient rule states that if (7.15) holds in all coordinate systems and for any vector $\boldsymbol{b}$ the resulting quantity $\boldsymbol{a}$ is a vector, then T_{ij} is a tensor.

Proof

The quotient rule is proved as follows: Since $\boldsymbol{a}$ is a vector,

$$a'_i = L_{ik}a_k = L_{ik}T_{kj}b_j.$$

Since $\boldsymbol{b}$ is a vector, it obeys $b_j = L_{mj}b'_m$ (note that this is the inverse transformation, from the dashed to the undashed frame, so the suffices of L are transposed). Substituting for b_j gives

$$a'_i = L_{ik}T_{kj}L_{mj}b'_m.$$

But since (7.15) holds in all coordinate systems,

$$a'_i = T'_{im}b'_m.$$

Subtracting these two results,

$$(T'_{im} - L_{ik}T_{kj}L_{mj})b'_m = 0.$$

If this result holds for any vector $\boldsymbol{b}$, then the quantity in brackets must be zero, so

$$T'_{im} = L_{ik}L_{mj}T_{kj}.$$

Therefore, T_{ij} is a second-rank tensor. □

A more general form of the quotient rule also holds: if an mth rank tensor a is linearly related to an nth rank tensor b through a quantity T with $m + n$ suffices, then T is a tensor of rank $m + n$.

EXERCISES

7.1 Show that the definition $L_{ij} = \boldsymbol{e}'_i \cdot \boldsymbol{e}_j$ is consistent with the matrix given in (7.3).

7.2 If $\boldsymbol{u}$ is a vector field, show that $\boldsymbol{\nabla} \cdot \boldsymbol{u}$ is a scalar field.

7.3 Given that $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors, show that the quantity $a_i b_j$ is a second-rank tensor.

7.4 Show that in a two-dimensional Cartesian coordinate system (x_1, x_2) the quantity

$$T_{ij} = \begin{pmatrix} x_1 x_2 & -x_1^2 \\ x_2^2 & -x_1 x_2 \end{pmatrix}$$

is a tensor.

7.5 If ϕ is a scalar field, show that the quantity

$$T_{jk} = \frac{\partial^2 \phi}{\partial x_j \partial x_k}$$

is a second-rank tensor.

7.6 If T_{ij} is a tensor, show that T_{ii} is a scalar.

7.7 Write the divergence theorem in the form of suffix notation and hence obtain the analogue of the divergence theorem for a second-rank tensor T_{ij}:

$$\iiint_V \frac{\partial T_{ij}}{\partial x_j}\, dV = \oiint_S T_{ij} n_j \, dS. \tag{7.16}$$

7.8 Write down the transformation rule for a tensor of rank four.

7.9 If Q_{ijkl} is a tensor of rank four, show that Q_{ijjl} is a tensor of rank two.

7.10 A quantity u_i has the property that for any vector $\boldsymbol{a}$, $u_i a_i$ is a scalar. Show that the u_i are the components of a vector. (This is a form of the quotient rule.)

7.3.2 Symmetric and anti-symmetric tensors

A second-rank tensor T_{ij} is said to be *symmetric* if $T_{ij} = T_{ji}$ and *anti-symmetric* if $T_{ij} = -T_{ji}$. A tensor of rank greater than two can be symmetric or anti-symmetric with respect to any pair of indices. For example δ_{ij} is a symmetric tensor, while ϵ_{ijk} is anti-symmetric with respect to any two of its indices.

It is important to verify that symmetry is a physical property of tensors, i.e. that if a tensor is symmetric in a Cartesian coordinate system it is also symmetric in other Cartesian coordinate systems. This can be confirmed as follows: suppose that A_{ij} is a symmetric tensor, so $A_{ij} = A_{ji}$. Then in a rotated frame,

$$A'_{ij} = L_{ik}L_{jm}A_{km} = L_{jm}L_{ik}A_{mk} = A'_{ji},$$

so A'_{ij} is also symmetric.

Example 7.7

Show that any second-rank tensor T_{ij} can be written as the sum of a symmetric tensor and an anti-symmetric tensor.

For any tensor T_{ij}, the tensor $S_{ij} = T_{ij} + T_{ji}$ is symmetric. Similarly, $A_{ij} = T_{ij} - T_{ji}$ is anti-symmetric. Since $S_{ij} + A_{ij} = 2T_{ij}$, T_{ij} can be written as $T_{ij} = S_{ij}/2 + A_{ij}/2$.

Example 7.8

The second-rank tensor T_{ij} obeys $\epsilon_{ijk}T_{jk} = 0$. Show that T_{ij} is a symmetric tensor.

By expanding out the implied double sum, for $i = 1$ we have $\epsilon_{123}T_{23} + \epsilon_{132}T_{32} = 0$, which gives $T_{23} = T_{32}$. Similarly the other required results follow from taking $i = 2$ and $i = 3$.

The same result may be obtained more elegantly by multiplying the given equation $\epsilon_{ijk}T_{jk} = 0$ by ϵ_{mni}:

$$\begin{aligned} 0 &= \epsilon_{mni}\epsilon_{ijk}T_{jk} \\ &= (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj})T_{jk} \\ &= T_{mn} - T_{nm}, \end{aligned}$$

so $T_{mn} = T_{nm}$.

7.3.3 Isotropic tensors

The two tensors δ_{ij} and ϵ_{ijk} have a special property. Their components are the same in all coordinate systems. A tensor with this property is said to be *isotropic*. Isotropic tensors are of great importance physically, and it turns out that there are very few examples of isotropic tensors. This is illustrated by the following results.

Theorem 7.1

There are no non-trivial isotropic first-rank tensors.

Proof

Suppose that there exists an isotropic first-rank tensor (i.e. an isotropic vector), $\boldsymbol{u} = (u_1, u_2, u_3)$. Now consider a rotation through $\pi/2$ about the x_3-axis, which is given by the matrix

$$L = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.17}$$

If $\boldsymbol{u}$ is a first-rank tensor then $u'_i = L_{ij}u_j = (u_2, -u_1, u_3)$. Now if $\boldsymbol{u}$ is isotropic, $u'_i = u_i$, so $u_1 = u_2$ and $u_2 = -u_1$. Therefore $u_1 = u_2 = 0$. By considering a rotation about the x_1-axis in a similar way, it can be shown also that $u_3 = 0$, so the only solution is $\boldsymbol{u} = (0, 0, 0)$. □

Theorem 7.2

The most general isotropic second-rank tensor is a multiple of δ_{ij}.

Proof

Suppose that a_{ij} is an isotropic second-rank tensor. Consider the rotation through $\pi/2$ about the x_3-axis given by (7.17). a_{ij} must obey $a'_{ij} = L_{im}L_{jn}a_{mn}$, which in terms of matrix multiplication is $a' = LaL^T$. Carrying out these matrix multiplications gives the result

$$LaL^T = \begin{pmatrix} a_{22} & -a_{21} & a_{23} \\ -a_{12} & a_{11} & -a_{13} \\ a_{32} & -a_{31} & a_{33} \end{pmatrix}. \tag{7.18}$$

This must be equal to a_{ij} if the tensor a_{ij} is isotropic. The terms on the diagonal give $a_{11} = a_{22}$. The other terms give $a_{13} = a_{23}$ and $a_{23} = -a_{13}$, from which $a_{13} = a_{23} = 0$. By considering the analogous rotations about the other coordinate axes it follows that $a_{11} = a_{22} = a_{33}$ and that all the off-diagonal terms are zero, so $a_{ij} = \lambda\delta_{ij}$, where λ is an arbitrary constant. □

Theorem 7.3

The most general isotropic third-rank tensor is a multiple of ϵ_{ijk}.

Proof

If a_{ijk} is an isotropic third-rank tensor, then

$$a_{ijk} = a'_{ijk} = L_{ip}L_{jq}L_{kr}a_{pqr}. \tag{7.19}$$

Consider the same rotation (7.17), for which the only non-zero elements of L are $L_{12} = 1$, $L_{21} = -1$ and $L_{33} = 1$. Therefore for any choice of i, j and k in (7.19), only one term on the r.h.s. is non-zero. Choosing $(i, j, k) = (1, 1, 1)$ gives $a_{111} = a_{222}$ and the choice $(i, j, k) = (2, 2, 2)$ gives $a_{222} = -a_{111}$, so $a_{111} = a_{222} = 0$. A different choice of rotation matrix would yield $a_{333} = 0$.

By making further choices of (i, j, k) the following equations can be obtained: $a_{112} = -a_{221}$, $a_{221} = a_{112}$, $a_{122} = a_{211}$, $a_{211} = -a_{122}$, $a_{121} = -a_{212}$, $a_{212} = a_{121}$. From these and the analogous equations involving the suffices 2 and 3 it follows that all 18 elements with two suffices equal are zero.

Finally, by considering the cases when i, j and k are all different, (7.19) gives $a_{123} = -a_{213}$, $a_{231} = -a_{132}$, $a_{312} = -a_{321}$. The analogous equations for rotations about the other axes can be used to show that $a_{123} = a_{231} = a_{312} = -a_{321} = -a_{132} = -a_{213}$, so that $a_{ijk} = \lambda\epsilon_{ijk}$ for some constant λ. □

Theorem 7.4

The most general isotropic fourth-rank tensor is

$$a_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}, \tag{7.20}$$

where λ, μ and ν are constants.

Proof

An isotropic fourth-rank tensor must obey

$$a_{ijkl} = L_{ip}L_{jq}L_{kr}L_{ls}a_{pqrs}. \tag{7.21}$$

Using the rotation (7.17), only one of the 81 terms in the implied sum on the r.h.s. is non-zero. Since $L_{12} = 1$, $L_{21} = -1$ and $L_{33} = 1$, a suffix 1 on the l.h.s. becomes a suffix 2 on the r.h.s., a suffix 2 on the l.h.s. becomes a suffix 1 on the r.h.s. and changes the sign, while a suffix 3 remains unchanged. By applying these rules, $a_{1113} = a_{2223} = -a_{1113}$, so $a_{1113} = a_{2223} = 0$. Similarly, any other term with three suffices equal and the fourth one different must be zero. Also $a_{2113} = -a_{1223} = -a_{2113}$ so $a_{2113} = a_{1223} = 0$ and all similar terms with only one pair of equal suffices are zero.

The only remaining terms are those with two pairs of equal suffices and those with all four suffices equal. Applying the rotation (7.17) to terms in which the first two suffices are equal and the last two suffices are equal gives $a_{1122} = a_{2211}$, $a_{1133} = a_{2233}$ and $a_{3322} = a_{3311}$. Using the rotations about the other coordinate axes it follows that these six terms are all equal. Similarly, $a_{1212} = a_{2121} = a_{1313} = a_{2323} = a_{3131} = a_{3232}$ and $a_{1221} = a_{2112} = a_{1331} = a_{2332} = a_{3113} = a_{3223}$. The terms with all four suffices equal must obey $a_{1111} = a_{2222} = a_{3333}$. Thus there can be at most four independent components of the tensor, a_{1122}, a_{1212}, a_{1221} and a_{1111}.

To proceed it is necessary to consider a different rotation, for example the rotation through an arbitrary angle θ about the x_3-axis given by

$$L = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.22}$$

Using this rotation, a_{1111} is related to all the terms with suffices equal to 1 or 2. Applying (7.21) gives

$$\begin{aligned} a_{1111} \;=\; & \cos^4\theta\, a_{1111} + \sin^4\theta\, a_{2222} \\ & + \sin^2\theta \cos^2\theta(a_{1122} + a_{2211} + a_{1212} + a_{2121} + a_{1221} + a_{2112}). \end{aligned}$$

Simplifying this equation and using the relations above, the trigonometric factors cancel out leaving

$$a_{1111} = a_{1122} + a_{1212} + a_{1221}, \tag{7.23}$$

so in fact there are only three independent components, which can be labelled $a_{1122} = \lambda$, $a_{1212} = \mu$, $a_{1221} = \nu$. The tensor a_{ijkl} can therefore be written in terms of λ, μ and ν in the form (7.20). Note that this ensures that (7.23) is satisfied.

□

7.4 Physical examples of tensors

Tensors appear in many contexts, including fluid mechanics, solid mechanics and general relativity. Some of these applications will be described in Chapter 8. The following two sections briefly consider two other examples of tensors.

7.4.1 Ohm's law

Ohm's law states that there is a linear relationship between the electric current $\boldsymbol{j}$ flowing through a material and the electric field $\boldsymbol{E}$ applied to the material. This can be written

$$\boldsymbol{j} = \sigma \boldsymbol{E}, \tag{7.24}$$

where the constant of proportionality σ is known as the conductivity (an inverse measure of electrical resistance). Note that (7.24) forces the vectors $\boldsymbol{j}$ and $\boldsymbol{E}$ to be parallel. For some materials, this may be true, but consider a substance with a layered structure made of different materials (Figure 7.2). For this material,

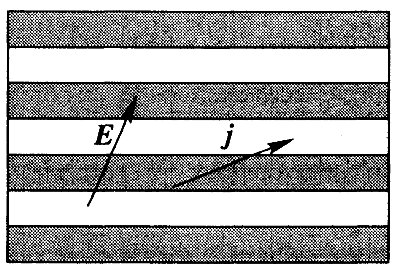

Fig. 7.2. For a material made up of layers, the electric field $\boldsymbol{E}$ and the electric current $\boldsymbol{j}$ may not be parallel.

current may flow more easily along the layers than across them. For example, if the substance is made of alternate layers of a conductor and an insulator, then current can only flow along the layers, regardless of the direction of the electric field.

It is useful therefore to have an alternative to (7.24) in which $\boldsymbol{j}$ and $\boldsymbol{E}$ do not have to be parallel. This can be achieved by introducing the ***conductivity tensor***, σ_{ik}, which relates $\boldsymbol{j}$ and $\boldsymbol{E}$ through the equation

$$j_i = \sigma_{ik} E_k. \tag{7.25}$$

Since $\boldsymbol{j}$ and $\boldsymbol{E}$ are vectors, it follows from the quotient rule that σ_{ik} is a tensor.

The values of σ_{ik} depend on the properties of the material. For example, suppose that there are alternating layers of a conductor with conductivity σ_0 and an insulator. If axes are chosen such that the x_3 direction is perpendicular to the layers, then in this coordinate system

$$\sigma_{ik} = \begin{pmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now suppose that the material has no such layered structure, so that there is no preferred direction and is made of a uniform material with conductivity σ_0. Such a material is said to be *isotropic*, meaning 'the same in all directions'. In this case $\sigma_{ik} = \sigma_0 \, \delta_{ik}$, so

$$j_i = \sigma_{ik} E_k = \sigma_0 \, \delta_{ik} E_k = \sigma_0 \, E_i$$

and so the simple rule

$$\boldsymbol{j} = \sigma_0 \, \boldsymbol{E}$$

holds. This is why δ_{ik} is said to be an isotropic tensor: it represents the relationship between two vectors that are always parallel, regardless of their direction.

7.4.2 The inertia tensor

Consider a body rotating with angular velocity $\boldsymbol{\Omega}$. Then, as shown in Section 1.3.1, the velocity vector at the position vector $\boldsymbol{r}$ is

$$\boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{r}.$$

The angular momentum of a particle of mass m is $\boldsymbol{h} = m\boldsymbol{r} \times \boldsymbol{v}$. The total angular momentum of a rotating body can then be determined as a volume integral, by considering dividing the body into small volume elements dV each with mass $\rho \, dV$, where ρ is the density of the body. The total angular momentum $\boldsymbol{H}$ is therefore given by

$$\begin{aligned} H_i &= \iiint_V \rho \, (\boldsymbol{r} \times \boldsymbol{v})_i \, dV \\ &= \iiint_V \rho \, (\boldsymbol{r} \times (\boldsymbol{\Omega} \times \boldsymbol{r}))_i \, dV \\ &= \iiint_V \rho \left(r^2 \Omega_i - (\boldsymbol{r} \cdot \boldsymbol{\Omega}) r_i \right) dV \end{aligned}$$

$$= \iiint_V \rho \left(r^2 \delta_{ij} \Omega_j - r_j \Omega_j r_i\right) \, dV$$
$$= \iiint_V \rho \left(r^2 \delta_{ij} - r_i r_j\right) \Omega_j \, dV.$$

Since Ω_j is a constant it can be taken out of the integral, leaving the equation

$$H_i = I_{ij} \Omega_j, \tag{7.26}$$

where I_{ij} is called the *inertia tensor* of the body and is defined by

$$I_{ij} = \iiint_V \rho \left(r^2 \delta_{ij} - r_i r_j\right) \, dV. \tag{7.27}$$

Note that as in the previous example, the tensor appears as a quantity relating two vectors, and the quotient rule confirms that I_{ij} is a tensor. The inertia tensor is an example of a symmetric tensor, since it is clear that $I_{ij} = I_{ji}$.

Example 7.9

Find the inertia tensor for a cube with sides of length $2a$ and constant density ρ, for rotations about its centre.

To find I_{ij} we need to compute two volume integrals. First,

$$\begin{aligned}
\iiint_V \rho r^2 \, dV &= \int_{-a}^{a} \int_{-a}^{a} \int_{-a}^{a} \rho \left(x^2 + y^2 + z^2\right) \, dx \, dy \, dz \\
&= 3\rho \int_{-a}^{a} \int_{-a}^{a} \int_{-a}^{a} x^2 \, dx \, dy \, dz \\
&= 3\rho \, (2a)(2a) \int_{-a}^{a} x^2 \, dx \\
&= 8\rho \, a^5 = M a^2,
\end{aligned}$$

where $M = 8\rho \, a^3$ is the mass of the cube. The second volume integral is

$$\iiint_V \rho \, r_i r_j \, dV.$$

For $i \neq j$ this is zero, since for example the integral of xy is zero since this is an odd function of x and y. For $i = j$, for example $i = j = 1$, we have

$$\iiint_V \rho x^2 \, dV = M a^2 / 3$$

from the working of the first integral. Putting the two parts together,

$$I_{ij} = M a^2 \delta_{ij} - M a^2 \delta_{ij} / 3 = \frac{2}{3} M a^2 \delta_{ij}.$$

Note that the inertia tensor is isotropic. This means that for a cube rotating about its centre, the rotation vector and angular momentum vector are always parallel.

Summary of Chapter 7

- Under a rotation of coordinate axes from a frame with unit vectors $\boldsymbol{e}_i$ to a frame with unit vectors $\boldsymbol{e}'_i$, the coordinates of a point are related by

$$x'_i = L_{ij}x_j$$

where $L_{ij} = \boldsymbol{e}'_i \cdot \boldsymbol{e}_j$.
- The inverse of the transformation is

$$x_i = L_{ji}x'_j,$$

so $L^{-1} = L^T$. Such a matrix is said to be *orthogonal.* In suffix notation, this result is written

$$L_{ij}L_{kj} = \delta_{ik}.$$

- A *scalar* s has the same value in each frame, $s' = s$.
- A *vector* v transforms according to the rule $v'_i = L_{ij}v_j$.
- If a quantity T_{ij} transforms according to the rule $T'_{ij} = L_{ik}L_{jm}T_{km}$ then T_{ij} is a *tensor* of second rank. The *rank* of a tensor is the number of free suffices. Thus vectors are tensors of rank one and scalars are tensors of rank zero.
- The quotient rule says that if $a_i = T_{ij}b_j$ and $\boldsymbol{a}$ is a vector for any choice of the vector $\boldsymbol{b}$, then T_{ij} is a tensor.
- A tensor T_{ij} is symmetric if $T_{ij} = T_{ji}$ and anti-symmetric if $T_{ij} = -T_{ji}$.
- δ_{ij} and ϵ_{ijk} are tensors of a special type known as isotropic tensors. This means that their components do not change when the coordinate axes are rotated. A second-rank isotropic tensor must be a multiple of δ_{ij} and a third-rank isotropic tensor must be a multiple of ϵ_{ijk}.
- In physical systems, tensors frequently arise as quantities relating two vectors. This allows two vectors to be linearly related to each other without being parallel. Examples include the conductivity tensor and the inertia tensor.

EXERCISES

7.11 B_{rs} is an anti-symmetric tensor, so $B_{rs} = -B_{sr}$. Show that the anti-symmetry persists in a rotated frame, i.e. $B'_{rs} = -B'_{sr}$.

7.12 If B_{rs} is an anti-symmetric tensor, show that $B_{rr} = 0$.

7.13 The third-rank tensor A_{ijk} is symmetric with respect to its first two suffices but anti-symmetric with respect to the second and third suffices. Show that all elements of A_{ijk} must be zero.

7.14 A quantity A_{ij} is related to a vector $\boldsymbol{B}$ by $A_{ij} = \epsilon_{ijk}B_k$.
(a) Show that A_{ij} is a tensor and describe its symmetry property.
(b) Find an equation for $\boldsymbol{B}$ in terms of A_{ij}.

7.15 Find an isotropic fourth-rank tensor that can be written in terms of ϵ_{ijk}.

7.16 Write down an isotropic fifth-rank tensor. Show that the most general isotropic fifth-rank tensor must have at least ten independent components.

7.17 Show that the kinetic energy E of a body rotating with angular velocity $\boldsymbol{\Omega}$ is related to its inertia tensor I_{jk} by $E = I_{jk}\Omega_j\Omega_k/2$.

8
Applications of Vector Calculus

This chapter provides a brief introduction to some of the many applications of vector calculus to physics. Each of these is a vast topic in itself and is the subject of numerous books and a great deal of current research, so it is not possible to go into any detail in this book. However, a number of important governing equations and results can be obtained using the methods described in the previous chapters. In particular, it will be seen that the equations describing the behaviour of physical quantities such as electric fields and the velocity of a fluid are written in terms of the gradient, divergence and curl operators.

The following sections discuss the flow of heat within a body, the behaviour of electric and magnetic fields, the mechanics of solids and the mechanics of fluids. There are however several other subjects which use the language of vector calculus, including the theories of quantum mechanics and general relativity.

8.1 Heat transfer

In this section the equation describing the flow of heat within a solid body is derived. The argument is based on the law of conservation of energy, so is similar to the argument for the conservation of mass for a fluid given in Section 5.1.1.

Consider a solid with a temperature T which depends on space and time and a thermal conductivity K. Then the heat flows from hot to cold at a rate proportional to the temperature gradient, so the heat flux $\boldsymbol{q}$ is given by $\boldsymbol{q} = -K\boldsymbol{\nabla}T$. The minus sign appears here because the vector $\boldsymbol{\nabla}T$ points in the direction of increasing temperature but the heat flows in the direction of decreasing temperature.

Now consider an arbitrary region within the solid, denoted by a volume V with surface S and outward normal $\boldsymbol{n}$. The thermal energy or heat content of a volume element dV is $T\,c\,\rho\,dV$ where ρ is the density of the material and c is its specific heat. So the total heat content H of the volume V is

$$H = \iiint_V T\,c\,\rho\,dV.$$

The rate of change of this heat content must equal the rate at which heat flows into the volume V, assuming that there are no sources of heat within V. This rate of inflow of heat is the integral of the heat flux $-\boldsymbol{q}\cdot\boldsymbol{n}$ over the surface S, where the minus sign appears since for heat to flow in, $\boldsymbol{q}$ must point in the opposite direction to $\boldsymbol{n}$. Equating the rate of change of heat content with the rate at which heat flows into V gives

$$\iiint_V \frac{\partial T}{\partial t}\,c\,\rho\,dV = \oiint_S -\boldsymbol{q}\cdot\boldsymbol{n}\,dS = \oiint_S K\boldsymbol{\nabla}T\cdot\boldsymbol{n}\,dS.$$

The surface integral on the r.h.s. can be converted to a volume integral using the divergence theorem, giving

$$\iiint_V \frac{\partial T}{\partial t}\,c\,\rho\,dV = \iiint_V \boldsymbol{\nabla}\cdot(K\boldsymbol{\nabla}T)\,dV. \tag{8.1}$$

Finally, since the volume V is arbitrary, the volume integrals can be cancelled, giving

$$c\,\rho\frac{\partial T}{\partial t} = \boldsymbol{\nabla}\cdot(K\boldsymbol{\nabla}T), \tag{8.2}$$

since if (8.2) were not true at any point in space, then introducing a small volume V around this point would contradict (8.1).