Solution Exam 2015

Vector functions are denoted by boldface. We write $\frac{\partial u_i}{\partial x_j}$ as $u_{i,j}$ and hence $\epsilon_{ij} = 1/2(u_{i,j} + u_{j,i})$.

Oppgave 1

a) There was an error in the formula sheet for Hooks law. Hence, we must accept derivations based on that error as well.

$$
\begin{aligned}
(\nabla \cdot \sigma)_i &= \sum_{j=1}^{3} \nabla_j \sigma_{ij} && (1)\\
&= \lambda \sum_{j=1}^{3} \nabla_j \sum_{k=1}^{3} \epsilon_{kk}\delta_{ij} + 2\mu \sum_{j=1}^{3} \nabla_j \epsilon_{ij} && (2)\\
&= \lambda \sum_{k=1}^{3} \nabla_i \nabla_k u_k + \mu \sum_{j=1}^{3} (\nabla_j \nabla_i u_j + \nabla_j \nabla_j u_i) && (3)\\
&= (\lambda + \mu)\, \nabla_i\, (\nabla \cdot \mathbf{u}) + \mu \nabla^2 u_i && (4)
\end{aligned}
$$

This can be written in vector form as

$$\nabla \cdot \sigma = \mu \nabla^2 \mathbf{u} + (\mu + \lambda)\nabla(\nabla \cdot \mathbf{u})$$

and hence, as $\nabla \cdot \sigma + \mathbf{f} = 0$, we obtain Navier's equation

$$\mu \nabla^2 \mathbf{u} + (\mu + \lambda)\nabla(\nabla \cdot \mathbf{u}) + \mathbf{f} = 0$$

b) Normal and tangent vectors:

$$\mathbf{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{t}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \mathbf{t}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Then

$$\sigma \cdot \mathbf{n} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{bmatrix}$$

Normal stress:

$$\mathbf{n}^T \sigma \mathbf{n} = \sigma_{33}$$

Shear:

$$\mathbf{t}_1^T \sigma \mathbf{n} = \sigma_{13}$$

$$\mathbf{t}_2^T \sigma \mathbf{n} = \sigma_{23}$$

Here

$$\begin{aligned}\sigma_{13} &= 2\mu\epsilon_{13} = \mu(u_{1,3} + u_{3,1}) \\ \sigma_{23} &= 2\mu\epsilon_{23} = \mu(u_{2,3} + u_{3,2}) \\ \sigma_{33} &= 2\mu\epsilon_{33} + \lambda\sum_k \epsilon_{kk} = 2\mu u_{3,3} + \lambda\sum_k u_{k,k}\end{aligned}$$

c) We have

$$\mathbf{u} = c\begin{pmatrix} -y \\ x \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

then

$$\epsilon(\mathbf{u}) = \begin{bmatrix} 0 & c-c \\ c-c & 0 \end{bmatrix}$$

As $\epsilon(\mathbf{u}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ we obtain that $\sigma = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Oppgave 2

a) Because of $\theta-$, $z-$, and $t-$ independence we have

$$\mathbf{v}(r,\theta,z,t) = \begin{bmatrix} u(r,\theta,z,t) \\ v(r,\theta,z,t) \\ w(r,\theta,z,t) \end{bmatrix} = \begin{bmatrix} u(r) \\ v(r) \\ w(r) \end{bmatrix}$$

Then the remaining terms of the continuity equation are:

$$\frac{1}{r}\frac{\partial(ru(r))}{\partial r} = 0$$

Hence, $ru(r)$ is constant and $u(r) = A/r$. Then $A = 0$ because $u(b) = 0$. The remaining terms of the θ-equation are

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v(r)}{\partial r}) - v/r^2 = 0$$

The solution of this equation is on the form:

$$v = A/r + Br$$

As $v(a) = 0$ and $v(b) = 0$, $A = B = 0$ and $v(r) = 0$.

b) We use the z equation to obtain a solution. The remaining parts of the $z-$ equation reads:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial w(r)}{\partial r}) = 0$$

We obtain:

$$r\frac{\partial w(r)}{\partial r} = A$$

Hence,

$$w(r) = A\ln(r) + B$$

At the boundaries, the no-slip condition holds. Therefore, we have that

$$w(b) = A\ln(a) + B = U$$

and

$$w(a) = A\ln(b) + B = 0$$

Solving for A and B, we find:

$$w(r) = U\frac{\ln(b/r)}{\ln(b/a)}$$

c) Similar as before, the z -equation reads in this case:

$$\eta\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial w(r)}{\partial r}) = \beta$$

$$\frac{\partial}{\partial r}(r\frac{\partial w(r)}{\partial r}) = \frac{\beta}{\eta}r$$

$$r\frac{\partial w(r)}{\partial r} = \frac{1}{2}\frac{\beta}{\eta}r^2 + A$$

The general solution is thus:

$$w(r) = \frac{1}{4}\frac{\beta}{\eta}r^2 + A\ln(r) + B.$$

The no-slip condition implies:

$$w(a) = U \quad w(b) = 0$$

which allows to solve for A and B, resulting into:

$$w(r) = -\frac{1}{4}\frac{\beta}{\eta}\left(b^2 - r^2\right) + \left(U + \frac{1}{4}\frac{\beta}{\eta}\left(b^2 - a^2\right)\right)\frac{\ln(b/r)}{\ln(b/a)}$$

d) The stress in a Newtonian fluid is given by:

$$\sigma = -pI + 2\mu S, \tag{5}$$

where S is the rate of strain and p the pressure in the fluid. For p, we have

$$p = \beta z + p_0, \tag{6}$$

where p_0 is some constant. From the formula sheet we find for S in cylindrical coordinates for the pressent flow field:

$$S_{rr} = S_{\theta\theta} = S_{zz} = S_{r\theta} = S_{z\theta} = 0 \tag{7}$$

$$S_{rz} = \frac{1}{2}\nabla_r w(r). \tag{8}$$

At the inner cylinder, we have $\mathbf{n} = \mathbf{e}_r$ and therefore:

$$\mathbf{T} = -p\mathbf{e}_r + \mu\nabla_r w(a)\mathbf{e}_z. \tag{9}$$

The normal stress is $-p$ and the shear stress $\mu\nabla_r w(a)$. At the outer cylinder, we have $\mathbf{n} = -\mathbf{e}_r$ and therefore:

$$\mathbf{T} = p\mathbf{e}_r - \mu\nabla_r w(b)\mathbf{e}_z. \tag{10}$$

As the normal is $\mathbf{n} = -\mathbf{e}_r$, the normal component is strictly speaking $-p$. We observe that the shear stress $-\mu\nabla_r w(b)$ is in oposite direction in order to satisfy the force balance. We have

$$\nabla_r w(r) = \frac{1}{2}\frac{\beta}{\eta}r + \frac{1}{r}\left(U + \frac{1}{4}\frac{\beta}{\eta}\left(b^2 - a^2\right)\right)\frac{1}{\ln(b/a)} \tag{11}$$

Oppgave 3

a) From the inverse Hook law we obtain:

$$\begin{aligned}
\epsilon_{xx} &= -\frac{\nu}{E}\sigma_0 && (12)\\
\epsilon_{yy} &= -\frac{\nu}{E}\sigma_0 && (13)\\
\epsilon_{zz} &= \frac{1}{E}\sigma_0 && (14)\\
\epsilon_{xy} &= 0 && (15)\\
\epsilon_{yz} &= 0 && (16)\\
\epsilon_{zx} &= 0 && (17)
\end{aligned}$$

The solution to these equations, given that there is no translation or rotation is given by

$$\begin{aligned}
u &= -\frac{\nu}{E}\sigma_0 x && (18)\\
v &= -\frac{\nu}{E}\sigma_0 y && (19)\\
w &= \frac{1}{E}\sigma_0 z. && (20)
\end{aligned}$$

The length after loading becomes:

$$L' = L + w(z = L) = L\left(1 - \frac{Mg}{E\pi R^2}\right) \tag{21}$$

and the radius:

$$R' = R\left(1 + \frac{\nu Mg}{E\pi R^2}\right) \tag{22}$$

As σ_{max} is the maximum stress, the material can support before yield, the mass M should satisfy:

$$M \leq \frac{\pi \sigma_{\text{max}} R^2}{g}, \tag{23}$$

For the present values we find:

$$M = 392\,\text{kg} \tag{24}$$

b) The Euler-Bernoulli equation for the present choice of coordinates is

$$-EI\frac{d^2u}{dz^2} = M_y(z), \tag{25}$$

which leads, given the definition of the moment, to:

$$EI\frac{d^2u}{dz^2} + Mgu = 0 \tag{26}$$

As the beam is simply supported, we have

$$u(0) = u(L) = 0, \tag{27}$$

which gives a condition for ω:

$$\omega_n = \frac{n\pi}{L}, \quad n = 1, 2, \ldots \tag{28}$$

Taking $u(z) = \sin(\omega_n z)$, we obtain

$$-EI\omega_n^2 + Mg = 0, \tag{29}$$

and therefore

$$M = \frac{EI}{gL^2} n^2 \pi^2 \quad n = 1, 2, \ldots, \tag{30}$$

which is smallest for $n = 1$:

$$M_{\text{min}} = \pi^2 \frac{EI}{gL^2} \tag{31}$$

The area moment of inertia can by computed by means of:

$$I = \int\limits_0^R \int\limits_0^{2\pi} r^3 \cos^2\phi \, dr d\phi = \frac{1}{4}\pi R^4. \tag{32}$$

This allows us to find a condition for R:

$$R = \left(\frac{4M_{\text{min}} g L^2}{\pi^3 E} \right)^{\frac{1}{4}} \tag{33}$$

Applying the numeric values, we find:

$$R = 7\,\text{mm}. \tag{34}$$